Linear Algebra MATH 50003
Problem Sheet 5

1. (i) Write down all the possible Jordan Canonical Forms of complex matrices having characteristic polynomial $x(x+1+i)^2(x-3)^3$.

(ii) Calculate the number of non-similar Jordan Canonical Forms over $\mathbb{F}_p$ (p prime) having characteristic polynomial $(x-1)^3(x+2)^5$. (Answer will depend on p.)

(iii) Calculate the number of similarity classes of 7×7 matrices over $\mathbb{C}$ with minimal polynomial $x^3(x-1)^2$.

2. Find the JCFs of the following matrices (i) over $\mathbb{C}$, and (ii) over $\mathbb{F}_p$ (p prime - answer may depend on p):

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

3. Among the following matrices, which pairs are similar?

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

4. You are given that A is an $n \times n$ matrix with $n \geq 6$ and the following properties:

$$c_A(x) = (x-1)^n, \ \operatorname{rank}(A-I) = n-3, \ \operatorname{rank}(A-I)^{n-4} = 1.$$

Find the JCF of A.

5. Let V be an n-dimensional vector space over a field F, and let $v_1, \ldots, v_n$ be basis of V. Define a linear map $T : V \to V$ by

$$T(v_i) = v_{i+1} \text{ for } 1 \leq i \leq n-1, \ \ T(v_n) = v_1$$

(ie. T maps $v_1 \to v_2 \to \cdots \to v_n \to v_1$).

(a) Show that if $F = \mathbb{C}$ then T has a JCF and find it.

(b) Let $F = \mathbb{R}$. For which values of n does T have a JCF over F?

(c) Let $n = p$, a prime, and let $F = \mathbb{F}_p$. Show that T has a JCF over F and find it.

6. (a) Prove that the Jordan block $J = J_n(\lambda)$ is similar to its transpose.

(b) Deduce from (a) that every square matrix over $\mathbb{C}$ is similar to its transpose.

(c) Show further that for any $n \times n$ matrix A over $\mathbb{C}$, there is an invertible *symmetric* matrix P such that $P^{-1}AP = A^T$.

7. (a) Let $0 \neq \lambda \in \mathbb{C}$. Show that $J_n(\lambda)^2$ is similar to $J_n(\lambda^2)$.

(b) Using the JCF theorem and part (a), prove that every invertible matrix A over $\mathbb{C}$ has a square root (i.e. show $\exists B$ such that $B^2 = A$).