Linear Algebra MATH 50003
Solutions to Problem Sheet 6

1. T sends $1 \to 0$, $x \to 1$, $x^2 \to 2x$, $x^3 \to 3x^2$, $x^4 \to 4x^3$.

For $F = \mathbb{C}$ or $\mathbb{F}_p$ with $p \neq 2, 3$, JCF is $J_5(0)$, Jordan basis is $1, x, \frac{1}{2}x^2, \frac{1}{6}x^3, \frac{1}{24}x^4$.

For $F = \mathbb{F}_3$, JCF is $J_3(0) \oplus J_2(0)$, Jordan basis is $1, x, 2x^2, x^3, x^4$.

For $F = \mathbb{F}_2$, JCF is $J_2(0)^2 \oplus J_1(0)$, Jordan basis is $1, x, x^2, x^3, x^4$.

2. (i) Char poly is $x - 2)^3$, min poly is $(x-2)^2$, so JCF is $J_2(2) \oplus J_1(2)$. To find Jordan basis: basis of $\text{Im}(A - 2I)$ is $u_1 = e_1 + e_2 - e_3$. To get Jordan basis of V, add a vector v_1 such that $(A - 2I)v_1 = u_1$, and a vector w_1 such that u_1, w_1 is a basis of $\ker(A - 2i)$. Take $v_1 = e_1$, $w_1 = e_2$. So Jordan basis is

$$e_1 + e_2 - e_3, e_1, e_2.$$

So matrix $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$.

(ii) Char poly $x(x-1)^3$, JCF is $J_3(1) \oplus J_1(0)$.

To find Jordan basis: primary decomp is $V = V_0 \oplus V_1$ where $V_0 = \ker(A) = \text{Sp}(e_1 - e_2)$, $V_1 = \ker(A - I)^3 = \text{Sp}(e_1 + e_3, e_2, e_4)$.

Find Jordan basis of V_1: basis of $(A - I)^2 V_1$ is e_2. Extend to Jordan basis of $(A - I)V_1$: $e_2, -e_4$. Extend to Jordan basis of V_1: $e_2, -e_4, -e_1 - e_3 - e_4$.

So Jordan basis of V is $e_2, -e_4, -e_1 - e_3 - e_4, e_1 - e_2$. Matrix P is $\begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$.

(iii) Char poly is $(x-1)^5$, JCF is $J_3(1) \oplus J_2(1)$, matrix $P = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}$.

3. (a) These are simple induction proofs; or you can just do part (ii) and sub in $r = 2$ or 3.

(b) Write $J_r(\lambda) = \lambda I + J$ where $J = J_r(0)$. We know the powers of J, by Prop 11.1. Since I and J commute, the Binomial theorem gives

$$J_r(\lambda)^n = (\lambda I + J)^n = \lambda^n I + n\lambda^{n-1} J + \binom{n}{2} \lambda^{n-2} J^2 + \cdots$$

Hence

$$J_r(\lambda)^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots & \binom{n}{r-1}\lambda^{n-r+1} \\ & \lambda^n & n\lambda^{n-1} & \cdots & \binom{n}{r-2}\lambda^{n-r+2} \\ & & \lambda^n & \cdots & \binom{n}{r-3}\lambda^{n-r+3} \\ & & & \ddots & \\ & & & & \lambda^n \end{pmatrix}$$

where of course a binomial coefficient $\binom{\alpha}{b}$ with $b > a$ is interpreted as 0.

(c) From Q2(i,ii) we have a matrix P such that $P^{-1}AP = J$, where J is a JCF. We know the formula for J^n by part (b). Hence we can work out $A^n = PJ^nP^{-1}$. I will leave you to compute this.

4. (i) A basis of Z is $e_1, T(e_1), T^2(e_1) = e_1, e_1 + e_3 + e_4, e_1 - e_2 + 2e_3 + 2e_4$.

(ii) $T^3(e_1) = -3T(e_1) + 3T^2(e_1)$, so from 12.1 of lecture notes, the char and min polys of T_Z are $x^3 - 3x^2 + 3x$. A basis of V/Z is the coset $Z + e_4$, and $\bar{T}(Z + e_4) = Z + e_4$, so the char and min poly of $\bar{T}$ is $x - 1$.

(iii) $Z(e_2, T)$ is equal to Z.

(iv) Yes, for example $Z(e_3, T) = V$.

5. (i) For $1 \leq r \leq n$ let $V_r = \mathrm{Sp}(e_1, \ldots, e_r)$. Then $V_r = Z(e_r, J)$ is J-invariant.

We claim that every J-invariant subspace is equal to V_r for some r. Let W be a J-invariant subspace, and suppose $W \neq 0$. Choose r maximal such that there exists $w \in W$ with $w = \sum_1^r \alpha_i e_i$ and $\alpha_r \neq 0$. Then $(J - \lambda I)^{r-1} w = \alpha_r e_1$, and hence $e_1 \in W$. Next, $(J - \lambda I)^{r-2} w = \alpha_{r-1} e_1 + \alpha_r e_2$, and so $e_2 \in W$. Continuing like this, we see that $e_i \in W$ for $i = 1, \ldots, r$, and hence $W = V_r$.

(ii) If $J = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_k}(\lambda)$, then the λ-eigernspace E_λ of J has dimension k. As F is infinite and $k \geq 2$, there are infinitely many subspaces of E_λ, and each of these is J-invariant.

The vector space V itself is J-invariant, and is not cyclic: this is because if V were cyclic, then by 12.1 of the notes, the minimal and characteristic polys of J would be the same, but this is not the case as $m_J(x) = (x - \lambda)^{max(n_i)}$.