Linear Algebra MATH 50003
Problem Sheet 6

1. Let V be the vector space of polynomials of degree at most 4 over a field F, where $F = \mathbb{C}$ or $\mathbb{F}_p$ (p prime). Define a linear map $T : V \to V$ by

$$T(f(x)) = f'(x) \ \ \forall f(x) \in V.$$

Find the JCF of T, and also a Jordan basis of V. (In the case where $F = \mathbb{F}_p$ the answer will depend on p.)

2. For each of the following matrices A over $\mathbb{C}$, find the JCF J of A, and find an invertible matrix P such that $P^{-1}AP = J$:

(i) $A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$.

3. (a) Let $\lambda \in \mathbb{C}$. Prove that for any integer $n \geq 1$,

$$J_2(\lambda)^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}, \quad J_3(\lambda)^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(b) Find a general formula for $J_r(\lambda)^n$. (Hint: write $J_r(\lambda) = \lambda I_r + J$, where $J = J_r(0)$ and try to use the Binomial Theorem.)

(c) Using (b), find A^n, where A is as in part (i) or (ii) of Q2.

4. Let $V = \mathbb{C}^4$, and let $T : V \to V$ be the linear map $T(v) = Av$ for all $v \in V$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find a basis of the cyclic subspace $Z = Z(e_1, T)$.

(ii) Find the characteristic and minimal polynomials of the restriction $T_Z : Z \to Z$ and of the quotient map $\bar{T} : V/Z \to V/Z$.

(iii) Find a vector $v \notin \mathrm{Sp}(e_1)$ such that $Z(v, T) = Z$.

(iv) Does there exist a vector $w \in V$ such that $Z(w, T) = V$?

5. Let F be a field, let $V = F^n$ and let $\lambda \in F$.

(i) Let $J = J_n(\lambda)$. Show that there are only a finite number of J-invariant subspaces of V, and that each of them is a cyclic subspace (ie. is of the form $Z(v, J)$ for some vector v).

(ii) Now let $J = J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_k}(\lambda)$, where $k \geq 2$ and $\sum_1^k n_i = n$. Show that if F is an infinite field, then there are infinitely many different J-invariant subspaces of V. Find one of these that is not a cyclic subspace.