Linear Algebra MATH 50003
Solutions to Problem Sheet 7

1. (a) (i) Only RCF is $C(x^2+1)^2 \oplus I_2$.

(ii) The polys x^2+x+1 and x^3+2 are irreducible over $\mathbb{Q}$. Only RCF is

$$C(x^2+x+1)^2 \oplus C(x^2+x+1) \oplus C(x^3+2)^2 \oplus C(x^3+2).$$

(iii) Factorization of x^5+1 into irreducibles in $\mathbb{F}_2[x]$ is $(x+1)(x^4+x^3+x^2+x+1)$. So only RCF is $C(x^4+x^3+x^2+x+1) \oplus I_4$.

(b) (i) Factorization of x^4+1 into irreducibles in $\mathbb{F}_3[x]$ is $(x^2+x-1)(x^2-x-1)$. There is no 5×5 matrix with this min poly.

(ii) Factorization of x^4+x^2+1 into irreducibles in $\mathbb{F}_3[x]$ is $(x+1)^2(x-1)^2$. So for example $J_2(-1) \oplus J_2(1) \oplus J_1(1)$ has this min poly.

2. (i) A satisfies the poly x^2+1, and since this is irreducible over $\mathbb{R}$, it is the minimal poly of A. Therefore the RCF of A is a block-diagonal sum of k copies of $C(x^2+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In particular $n = 2k$ is even.

(ii) By (i), A is similar over $\mathbb{R}$ to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (k copies). This is similar to $\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$.

3. The first matrix, call it A, has characteristic poly $x^4-4x^2+4 = (x^2-2)^2$. Check $A^2-2I \neq 0$, so $m_A(x) = (x^2-2)^2$ also. So the RCF of A is $C(x^4-4x^2+4)$.

Call the second matrix B. Check that $c_B(x) = (x^2+1)^4$, $m_B(x) = (x^2+1)^2$. So, writing $f = x^2+1$, the RCF of B is either $C(f^2) \oplus C(f^2)$ or $C(f^2) \oplus C(f) \oplus C(f)$. Now check that $f(B) = B^2+I$ has rank 4. Hence the RCF is $C(f^2) \oplus C(f^2)$.

4. All these matrices have char poly $x^2(x+1)^2(x^2+x+1)$. Call them $A_1, \ldots, A_4$, respectively.

Check that $\text{rank}(A_1) = 5$, $\text{rank}(A_1+I) = 4$, so the geometric mults of 0 and 1 for A_1 are 1 and 2. Therefore the RCF of A_1 is $C(x^2) \oplus I_2 \oplus C(x^2+x+1)$.

Next, $\text{rank}(A_2) = 4$, $\text{rank}(A_2+I) = 4$, so the RCF of A_2 is $0_2 \oplus I_2 \oplus C(x^2+x+1)$.

Next, $\text{rank}(A_3) = 4$, $\text{rank}(A_3+I) = 5$, so the RCF of A_3 is $0_2 \oplus C(x+1)^2 \oplus C(x^2+x+1)$.

Finally, $\text{rank}(A_4) = 4$, $\text{rank}(A_3+I) = 4$, so the RCF of A_4 is $0_2 \oplus I_2 \oplus C(x^2+x+1)$.

We conclude using the RCF Theorem that the only similar pair are A_2, A_4.

4. This is tricky. (At least I can't think of an easy, quick proof.)

Let V be a vector space over F, and $T : V \to V$ a linear map such that there is a basis B with $[T]_B = C(f) \oplus C(g)$. Let $r = \deg(f), s = \deg(g)$, so $n = r+s$. Then there are vectors v, w such that

$$B = \{v, T(v), \ldots, T^{r-1}(v),\, w, T(w), \ldots, T^{s-1}(w)\}.$$

Then $V = V_1 \oplus V_2$, where $V_1 = \mathrm{Sp}(v, T(v), \ldots, T^{r-1}(v))$, $V_2 = \mathrm{Sp}(w, T(w), \ldots, T^{s-1}(w))$, and moreover the T-annihilators of v and w are the polys $f(x)$ and $g(x)$, respectively.

Now consider the vector $u = v + w$. For a polynomial $p(x)$, we have $p(T)(u) = p(T)(v) + p(T)(w) \in V_1 \oplus V_2$. If $p(T)(u) = 0$ then $p(T)(v) = p(T)(w) = 0$, and hence $p(x)$ is divisible by both annihilators $f(x)$ and $g(x)$. As these are coprime, it follows that $f(x)g(x)$ divides $p(x)$. Hence the T-annihilator of u is $f(x)g(x)$, which has degree $r+s = n$. It follows that $u, T(u), \ldots, T^{n-1}(u)$ is a basis of V, and the matrix of T with respect to this basis is $C(fg)$. Hence $C(f) \oplus C(g) \sim C(fg)$, as required.

5. By the RCF Theorem 12.6, if $m_A(x) = \prod_{i=1}^{t} f_i(x)^{k_i}$, where $f_1, \ldots, f_t$ are distinct irreducible polys, then

$$A \sim \bigoplus_{j=1}^{r_1} C(f_1^{k_{1j}}) \oplus \cdots \oplus \bigoplus_{j=1}^{r_t} C(f_t^{k_{tj}}), \qquad (1)$$

for unique integers $k_i = k_{i1} \geq \cdots \geq k_{ir_i}$ for $1 \leq i \leq t$.

By Q4, there is a unique choice of monic polys $g_1, g_2, \ldots g_r \in F[x]$ such that $A \sim \bigoplus_{j=1}^{r} C(g_j)$ and $g_{i+1}|g_i$ for all i: namely, take g_1 to be the product of all the largest powers of $f_1, \ldots, f_t$ occurring in (1) (ie. $g_1 = f_1^{k_{11}} \cdots f_t^{k_{t1}}$ (note this is equal to $m_A(x)$); then take g_2 to be the product of the remaining largest powers of $f_1, \ldots, f_t$; and so on, until all the factors in (1) are used up.

In Q1(a)(i), we have $g_1 = (x^2+1)^2(x-1)$, $g_2 = x - 1$.

In Q1(a)(ii), $g_1 = (x^2+x+1)^2(x^3+2)^2$, $g_2 = (x^2+x+1)(x^3+2)$.

In 1(a)(iii), $g_1 = x^5 + 1$, $g_2 = g_3 = g_4 = x + 1$.

6. A routine if tedious check shows that the number of irreducible monic polys over $\mathbb{F}_3$ of degrees 1,2,3 are 2,3,8 respectively.

The number of RCFs with char poly $(x+1)^3$ or $(x-1)^3$ is 3 of each, for a total of 6.

The number of RCFs with char poly $(x+1)(x-1)^2$ or $(x+1)^2(x-1)$ is 2 of each, total of 4.

Each char poly $(x+1)q(x)$ or $(x-1)q(x)$ (q a quadratic irred) has one RCF: total of 6.

Each char poly $c(x)$, and irred cubic, has one RCF: total 8.

So the total number of conjugacy classes in $GL(3, \mathbb{F}_3)$ is $6 + 4 + 6 + 8 = 24$.

Similar reasoning for $GL(4, \mathbb{F}_2)$ gives a total of 14 conjugacy classes.