Linear Algebra MATH 50003
Problem Sheet 7

1. (a) Let $M_n(F)$ be the set of all $n \times n$ matrices over a field F. Find all possible Rational Canonical Forms for

(i) matrices in $M_6(\mathbb{R})$ with minimal polynomial $(x^2+1)^2(x-1)$

(ii) matrices in $M_{15}(\mathbb{Q})$ with minimal polynomial $(x^2+x+1)^2(x^3+2)^2$

(iii) matrices in $M_8(\mathbb{F}_2)$ with minimal polynomial x^5+1.

(b) True or false:

(i) there exists $A \in M_5(\mathbb{F}_3)$ with minimal polynomial x^4+1

(ii) there exists $A \in M_5(\mathbb{F}_3)$ with minimal polynomial x^4+x^2+1.

2. Let A be a real $n \times n$ matrix such that $A^2 + I_n = 0$.

(i) Prove that n is even.

(ii) Prove that A is similar over $\mathbb{R}$ to the matrix $\begin{pmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{pmatrix}$.

3. (a) Find the Rational Canonical Forms of the following matrices over $\mathbb{Q}$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

(b) Which pairs among the following matrices over $\mathbb{F}_2$ are similar to each other:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

4. Let F be a field, and let $f, g \in F[x]$ be coprime polynomials (ie. $\gcd(f,g) = 1$). Prove that the companion matrix $C(fg)$ is similar over F to the block-diagonal matrix $C(f) \oplus C(g)$.

5. Using the RCF Theorem 12.6 of lectures, together with Q4, deduce another version of the RCF Theorem: if A is $n \times n$ over F, then there are unique monic polynomials $g_1, \ldots, g_r \in F[x]$ such that

(i) A is similar to $C(g_1) \oplus \cdots \oplus C(g_r)$, and

(ii) g_{i+1} divides g_i for all $i = 1, \ldots, r-1$.

Express all the RCFs you found in Q1(a) in this form.

6. Calculate the number of conjugacy classes in the general linear groups $GL(3, \mathbb{F}_3)$ and $GL(4, \mathbb{F}_2)$.