Linear Algebra MATH 50003
Solutions to Problem Sheet 8

1. (a) Clearly f_v is linear, so $f_v \in V^*$. The map $v \to f_v$ is a linear map $\pi : V \to V^*$, and $v \in \ker(\pi)$ implies $v^T w = 0\ \forall w \in V$, which implies $v = 0$. Hence π is injective, and since we know that $\dim V = \dim V^*$, π is also surjective, proving (a).

(b) $w_1 = (-3, -5, -2)^T$, $w_2 = (2, 1, 0)^T$, $w_3 = (1, 2, 1)^T$.

2. (a) π_v is clearly linear, so $\pi_v \in V^{**}$. The map $v \to \pi_v$ from $V \to V^{**}$ is linear, and if v is in the kernel then $\pi_v(f) = f(v) = 0$ for all $f \in V^*$, which implies $f = 0$. Hence the map is injective, and as $\dim V = \dim V^* = \dim V^{**}$, it is an isomorphism.

(b) (i) $f \in (U+W)^0 \Rightarrow f \in U^0$ and $f \in W^0 \Rightarrow f \in U^0 \cap W^0$, so $LHS \subseteq RHS$. Also

$$f \in U^0 \cap W^0 \Rightarrow f(u) = f(w) = 0 \forall u \in U, w \in W \Rightarrow f(u+w) = 0 \forall u, w \Rightarrow f \in (U+W)^0,$$

so $RHS \subseteq LHS$.

(ii) Strangely, this does not seem to be as easy as (i) and requires a dimension argument. First, $f \in U^0 + W^0 \Rightarrow f = f_1 + f_2$ with $f_1 \in U^0, f_2 \in W^0$, so for $v \in U \cap W$, we have $f(v) = f_1(v) + f_2(v) = 0$. Hence $U^0 + W^0 \subseteq (U \cap W)^0$.

For the reverse inclusion, we show the two sides have the same dimension. Let $n = \dim V$ and use Prop. 13.2:

$$\begin{aligned}\dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\ &= (n - \dim U) + (n - \dim W) - (n - \dim(U + W)) \text{ (using part (i))} \\ &= n - (\dim U + \dim W - \dim(U + W)) \\ &= n - \dim U \cap W \\ &= \dim(U \cap W)^0.\end{aligned}$$

3. Clealry ϕ_1, ϕ_2, ϕ_3 are in V^*, and send the basis vectors $1, x, x^2$ of V as follows:

$$\begin{aligned}(\phi_1(1),\ \phi_1(x),\ \phi_1(x^2)) &= (1, \tfrac{1}{2}, \tfrac{1}{3}), \\ (\phi_2(1),\ \phi_2(x),\ \phi_2(x^2)) &= (0, 1, 2) \\ (\phi_3(1),\ \phi_3(x),\ \phi_3(x^2)) &= (1, 0, 0).\end{aligned}$$

The three vectors on the RHS are linearly independent, hence ϕ_1, ϕ_2, ϕ_3 is a basis of V^*. Computing the dual basis to these vectors, we find that the basis of V dual to ϕ_1, ϕ_2, ϕ_3 is f_1, f_2, f_3, where

$$f_1(x) = 3x - \frac{3}{2}x^2,\ f_2(x) = -\frac{1}{2}x + \frac{3}{4}x^2,\ f_3(x) = 1 - 3x + \frac{3}{2}x^2.$$

4. (a) Let $e_1, \ldots, e_n$ be the standard basis of F^n, and define $a_{ij} = (e_i, e_j)$, and $A = (a_{ij})$. If $u = \sum u_i e_i, v = \sum v_i e_i \in V$, then using the inner product axioms,

$$(u, v) = \sum_{i,j} u_i \bar{v}_j (e_i, e_j) = \sum_{i,j} u_i a_{ij} \bar{v}_j = u^T A \bar{v}.$$

As seen in lecture notes, A is Hermitian and positive definite.

(b) The definition $(u, v) = u^T A \bar{v}$ satisfies the inner product axioms (1) and (2), and for $v \neq 0$ we have $(v, v) = v^T A \bar{v} > 0$ as A is positive definite, so axiom (3) also holds.

(c) (i) This does not satify the left-linearity axiom (1), eg. for $u = (1,0)^T$, we have $(u,u) = 4$ but $(iu,u) = 0$.

(ii) This is $(u,v) = u^T A\bar{v}$ where $A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$. This matrix is Hermitian, but is not positive definite as it has an evalue 0: for $u = (-i,1)^T$ we have $A\bar{u} = 0$, so $(u,u) = u^T A\bar{u} = 0$, contradicting axiom (3).

(iii) This is $(u,v) = u^T A\bar{v}$ where $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. This matrix is Hermitian and is also positive definite, since

$$(u,u) = |u_1|^2 - u_1\bar{u}_2 - \bar{u}_1 u_2 + 2|u_2|^2 = |u_1 - u_2|^2 + |u_2|^2.$$

5. (i) $(u,v) = (u,w)\ \forall u \Rightarrow (u, v-w) = 0\ \forall u \Rightarrow (v-w, v-w) = 0 \Rightarrow v - w = 0$.

(ii) $||u+v||^2 = (u+v, u+v) = (u,u) + (u,v) + (v,u) + (v,v) = ||u||^2 + ||v||^2$.

(iii) $||u+v||^2 = ||u||^2 + ||v||^2 + (u,v) + \overline{(u,v)} \le ||u||^2 + ||v||^2 + 2|(u,v)| \le ||u||^2 + ||v||^2 + 2||u||\,||v||$ (by Cauchy-Schwarz) $= (||u|| + ||v||)^2$.

(iv) Suppose $\sum_1^r \lambda_i v_i = 0$. Then $0 = (\sum \lambda_i v_i, v_j) = \lambda_j (v_j, v_j)$. Hence (as $v_j \neq 0$), $\lambda_j = 0$ for all j, and so $v_1, \ldots, v_r$ are linearly indep.

(v) $(u-v, u-v) = ||u||^2 + ||v||^2 - (u,v) - (v,u) = 1 + 1 - 1 - 1 = 0$, hence $u - v = 0$.

(vi) For $w \in W, x \in W^\perp$ we have $(w,x) = 0$, hence $W \subseteq (W^\perp)^\perp$. Also $\dim(W^\perp)^\perp = \dim V - \dim W^\perp$ (by Prop 14.4) $= \dim V - (\dim V - \dim W) = \dim W$. Hence $W = (W^\perp)^\perp$.

6. (a) Orthonormal basis u_1, u_2, u_3 where $u_1 = 1$, $u_2 = \sqrt{3}(1-2x)$, $u_3 = \sqrt{5}(-1 + 6x - 6x^2)$.

(b) ϕ sends $u_1 \to 1$, $u_2 \to \sqrt{3}$, $u_3 \to -\sqrt{5}$. So take $v = u_1 + \sqrt{3}u_2 - \sqrt{5}u_3 = 9 - 36x + 30x^2$.

7. (a) (i) Let $u = (a_1, \ldots, a_n)$, $v = (1, \ldots, 1)$. By Cauchy-Schwarz, using the usual dot product on $\mathbb{R}^n$,

$$|(u,v)|^2 \le ||u||^2 ||v||^2 \Rightarrow \left(\sum a_i\right)^2 \le \left(\sum a_i^2\right) n \Rightarrow \sum a_i^2 \ge \frac{1}{n}.$$

(ii) Let $u = \left(\frac{1}{\sqrt{a_1}}, \ldots, \frac{1}{\sqrt{a_n}}\right)$, $v = (\sqrt{a_1}, \ldots, \sqrt{a_n})$. Then

$$n^2 = |(u,v)|^2 \le ||u||^2 ||v||^2 = \sum \frac{1}{a_i}.$$

(b) The cubes have total surface area $6(a^2 + b^2 + c^2)$, and the cuboids $6(ab + bc + ca)$. If we take $u = (a,b,c)$, $v = (b,c,a)$, Cauchy-Schwarz gives $ab + bc + ca < a^2 + b^2 + c^2$ (strict inequality as u, v are not scalar multiples of each other).