Linear Algebra MATH 50003 Problem Sheet 8

1. (a) Let F be a field and $V = F^n$ the n-dimensional vector space of column vectors over F. For $v \in V$ define $f_v : V \to F$ by

$$f_v(w) = v^T w \quad \forall w \in V.$$

Prove that $f_v \in V^*$, and that $V^* = \{f_v : v \in V\}$.

(b) Let $V = \mathbb{R}^3$ with basis $\{v_1, v_2, v_3\}$, where

$$v_1 = (1, -2, 3)^T,\ v_2 = (1, -1, 1)^T,\ v_3 = (2, -4, 7)^T.$$

Find $w_1, w_2, w_3 \in V$ such that $f_{w_1}, f_{w_2}, f_{w_3}$ is the dual basis of V^*.

2. Let V be a finite-dimensional vector space over a field F. Denote the dual space of V by V^*, and the dual of V^* by V^{**}.

(a) For $v \in V$ define $\pi_v : V^* \to F$ by $\pi_v(f) = f(v)$ for all $f \in V^*$. Show that $\pi_v \in V^{**}$, and the map $v \to \pi_v$ is an isomorphism $V \to V^{**}$.

(b) For subspaces U, W of V, prove that

(i) $(U + W)^0 = U^0 \cap W^0$, and

(ii) $(U \cap W)^0 = U^0 + W^0$.

3. Let V be the vector space over $\mathbb{R}$ of polynomials of degree at most 2, and define $\phi_1, \phi_2, \phi_3 : V \to \mathbb{R}$ by

$$\phi_1(p(x)) = \int_0^1 p(x)dx,\ \phi_2(p(x)) = p'(1),\ \phi_3(p(x)) = p(0)$$

for all $p(x) \in V$. Show that $\{\phi_1, \phi_2, \phi_3\}$ is a basis of V^*, and find the basis of V that is dual to this basis.

4. (a) Let $F = \mathbb{R}$ or $\mathbb{C}$, let $V = F^n$, and let $(,)$ be an inner product on V. Show that there exists a positive definite Hermitian matrix A such that $(u, v) = u^T A\bar{v}$ for all $u, v \in V$.

(b) Show conversely that the definition of (u, v) in (a) does define an inner product on V.

(c) Which of the following expressions defines an inner product on $\mathbb{C}^2$ (where $u = (u_1, u_2)^T$ etc.):

(i) $(u, v) = (u_1 + \bar{u}_1)(v_1 + \bar{v}_1) + (u_2 + \bar{u}_2)(v_2 + \bar{v}_2)$

(ii) $(u, v) = u_1\bar{v}_1 - iu_1\bar{v}_2 + iu_2\bar{v}_1 + u_2\bar{v}_2$

(iii) $(u, v) = u_1\bar{v}_1 - u_1\bar{v}_2 - u_2\bar{v}_1 + 2u_2\bar{v}_2$.

5. Let V be a finite-dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. Prove the following statements:

(i) if $v, w \in V$ are such that $(u, v) = (u, w)$ for all $u \in V$, then $v = w$.

(ii) Pythagoras: if $(u, v) = 0$, then $||u + v||^2 = ||u||^2 + ||v||^2$.

(iii) Parts 2 and 3 of Prop. 14.1: if $u, v, w \in V$ then $||u + v|| \leq ||u|| + ||v||$, and $||u - v|| \leq ||u - w|| + ||w - v||$.

(iv) Any orthogonal set $v_1, \ldots, v_r$ of nonzero vectors is linearly independent.

(v) If $||u|| = ||v|| = (u, v) = 1$, then $u = v$.

(vi) If W is a subspace of V, then $(W^\perp)^\perp = W$.

6. Let V be the vector space over $\mathbb{R}$ of polynomials of degree at most 2, with inner product defined by

$$(f, g) = \int_0^1 f(x)g(x)dx \quad \forall f, g \in V.$$

(a) Starting with the basis $1, x, x^2$ and using Gram-Schmidt, find an orthonormal basis of V.

(b) Define $\phi \in V^*$ by $\phi(f(x)) = f(0)$ for all $f \in V$. Find $v \in V$ such that

$$\phi(f) = (f, v) \quad \forall f \in V.$$

7. (a) Let $a_1, \ldots, a_n$ be positive real numbers such that $\sum_1^n a_i = 1$. Use Cauchy-Schwarz to prove the following:

(i) $\sum_1^n a_i^2 \geq \frac{1}{n}$

(ii) $\sum_1^n \frac{1}{a_i} \geq n^2$.

(b) Let a, b, c be real numbers with $0 < a < b < c$. Which of the following has the greater total surface area:

(i) three cubes, one of side a, one of side b and one of side c, or

(ii) three identical cuboids, each with sides a, b, c?