Linear Algebra MATH 50003
Solutions to Problem Sheet 9

1. (a) (i) Let $w_0 = \pi_W(v)$, so that $v = w_0 + w'$ with $w' \in W^\perp$. Then for any $w \in W$,

$$\begin{aligned} ||v - w_0||^2 &\leq ||v - w_0||^2 + ||w_0 - w||^2 \\ &= ||(v - w_0) + (w_0 - w)||^2 \text{ (by Pythagoras, as } (v - w_0, w_0 - w) = 0) \\ &= ||v - w||^2. \end{aligned}$$

Hence $||v - w||$ is minimal for $w = w_0$. Also equality holds in the above iff $||w_0 - w||^2 = 0$, ie. iff $w = w_0$, so w_0 is the unique closest vector to v.

(ii) Extend $v_1, \ldots, v_r$ to an orthonormal basis $v_1, \ldots, v_n$ of V. By 14.6 of lectures,

$$v = \sum_{i=1}^{r} (v, v_i)\, v_i + \sum_{i=s+1}^{n} (v, v_i)\, v_i.$$

The first sum is in W and the second is in $W^\perp$. Hence $\pi_W(v)$ is equal to the first sum.

(b) (i) Use Gram-Schmidt to get an orthonormal basis $\{u_1, u_2\}$ of W: $u_1 = \frac{1}{\sqrt{2}}(1,1,0,0)$, $u_2 = \frac{1}{\sqrt{5}}(0,0,1,2)$. Now let $v = (1,2,3,4)$ and compute

$$\pi_W(v) = (v, u_1)\, u_1 + (v, u_2)\, u_2 = \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5} \right).$$

(ii) let V be the vector space of real polys of degree ≤ 3 with inner product $(f, g) = \int_0^1 f(x)g(x)dx$, and let W the subspace $\{p(x) \in V : p(0) = p'(0) = 0\}$, so that $W = \mathrm{Sp}(x^2, x^3)$. We are looking for the closest point in W to the poly $q(x) = 2 + 3x$, ie. $\pi_W(q(x))$.

Use Gram-Schmidt to find an orthomormal basis of W:

$$f_1 = \sqrt{5}x^2, \ f_2 = \sqrt{7}(-5x^2 + 6x^3).$$

Then compute that

$$\pi_W(q(x)) = (q, f_1)u_1 + (q, f_2)f_2 = 24x^2 - \frac{203}{10}x^3.$$

2. This is all quite routine using the equation $(T(u), v) = (u, T^*(v))$.

(i) By defn of $(S+T)^*$, $(u, (S+T)^*(v)) = ((S+T)(u), v) = (S(u) + T(u), v) = (S(u), v) + (T(u), v) = (u, S^*(v)) + (u, T^*(v)) = (u, S^*(v) + T^*(v))$. Hence $(S+T)^*(v) = S^*(v) + T^*(v)$ for all v.

(ii) $(u, (\lambda T)^*(v)) = (\lambda T(u), v) = \lambda(T(u), v) = \lambda(u, T^*(v)) = (u, \bar{\lambda} T^*(v))$. Hence $(\lambda T)^*(v) = \bar{\lambda} T^*(v)$ for all v.

(iii) $(u, (T^*)^* v) = (T^*(u), v) = \overline{(v, T^*(u))} = \overline{(T(v), u)} = (u, T(v))$. Hence $(T^*)^*(v) = T(v)$ for all v.

(iv) $((ST)^* u, v) = (u, ST(v)) = (S^*(u), T(v)) = (T^* S^*(u), v)$, hence $(ST)^* = T^* S^*$.

(v) $v \in \ker(T^*) \Leftrightarrow (u, T^*(v)) = 0 \ \forall u \Leftrightarrow (T(u), v) = 0 \ \forall u \Leftrightarrow v \in \mathrm{Im}(T)^\perp$.

(vi) By (iii) and (v), $\mathrm{Im}(T^*)^\perp = \ker(T^{**}) = \ker(T)$, hence $\ker(T)^\perp = \mathrm{Im}(T^*)^{\perp\perp} = \mathrm{Im}(T^*)$.

3. if $c_1, \dots, c_n$ are the columns of P, then the ij-entry of $P^T \bar{P}$ is equal to $c_i^T \bar{c}_j$, so $P^T \bar{=} I$ iff $c_1, \dots, c_n$ is an orthonormal set in $\mathbb{C}^n$.

Let λ be an evalue of P, and $v \in \mathbb{C}^n$ a unit evector with $Pv = \lambda v$. Then

$$\lambda\bar{\lambda} = \lambda\bar{\lambda}v^T\bar{v} = (\lambda v)^T(\bar{\lambda}\bar{v}) = (Pv)^T(\bar{P}\bar{v}) = v^T P^T \bar{P}\bar{v} = v^T\bar{v} = 1.$$

Hence $|\lambda| = 1$.

As the first col is a unit vector, $y = \pm 1$. Then the second column being unit implies $x = 0$. The first and third cols are orthogonal, which gives $y + z + i(y-1) = 0$. Hence $y = 1, z = -1, x = 0$.

For any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the matrix λI_n is unitary. There are infinitely many of these.

On the other hand any diagonal orthogonal matrix must have ± 1's on the diagonal, so there are only 2^n of these.

4. 1st matrix: evalues 1,3, unitary $P = \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$.

2nd matrix: evalues 1,2,3, unitary $P = \frac{1}{2}\begin{pmatrix} -\sqrt{2}i & 0 & \sqrt{2}i \\ -1 & \sqrt{2} & -1 \\ 1 & \sqrt{2} & 1 \end{pmatrix}$.

3rd matrix: evalues $2, 2, -2$, unitary $P = \frac{1}{2}\begin{pmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{pmatrix}$.

(In each case, many other P's are possible.)

5. (a) As T is self-adjoint, $(T(v), v) = (v, T^*(v)) = (v, T(v)) = \overline{(T(v), v)}$ for all $v \in V$. Hence $(T(v), v) \in \mathbb{R}$.

(b) We know by the spectral theorem 15.3 that V has an orthonormal basis $B = \{v_1, \dots, v_n\}$ of T-eigenvectors, with corresponding evalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct, of course).

($\Rightarrow$) Suppose T is positive. Then for each i, we have

$$0 < (T(v_i), v_i) = \lambda_i(v_i, v_i) = \lambda_i ||v_i||^2,$$

and hence $\lambda_i > 0$.

($\Leftarrow$) Suppose $\lambda_i > 0$ for all i. Let $0 \neq v \in V$, so $v = \sum_{j=1}^n \alpha_j v_j$ for some scalars α_j. Then $T(v) = \sum_{j=1}^n \alpha_j \lambda_j v_j$, so

$$\begin{aligned} (T(v), v) &= \left(\textstyle\sum_j \alpha_j\lambda_j v_j, \sum_k \alpha_k v_k\right) \\ &= \textstyle\sum_j \lambda_j \alpha_j \bar{\alpha}_j \\ &= \textstyle\sum_j \lambda_j |\alpha_j|^2 \\ &> 0. \end{aligned}$$

Hence T is positive.

(c) The 1st and 2nd matrices in Q4 have positive evalues, so are positive maps.

(d) Suppose T is positive. By (b), all the evalues $\lambda_i > 0$. So we can define a linear map $S : V \to V$ by taking

$$S(v_i) = \sqrt{\lambda_i}\, v_i \quad \text{for all } i.$$

Then $S^2(v_i) = \lambda_i v_i = T(v_i)$ for all i, so $S^2 = T$. Also the matrix $[S]_B$ is a real diagonal matrix, so is symmetric, and hence S is self-adjoint by 15.2 of lecture notes.

(e) We have $P^{-1}AP = \text{diag}(\lambda_1, \ldots, \lambda_n)$, so a square root of A is PDP^{-1}, where $D = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. We worked out the evalues λ_i and the matrices P in Q4, so can work out the square roots. I'll omit the calcs.

6. (i) $||S(v)||^2 = (S(v), S(v)) = (v, v) = ||v||^2$.

(ii) For $u, v \in V$, $(u, v) = (S(u), S(v)) = (u, S^*S(v))$. Hence $S^*S(v) = v$ for all v, and so $S^*S = I_V$. This implies $S^* = S^{-1}$, and so also $SS^* = I_V$.

(iii) $(S^*(u), S^*(v)) = (u, S^{**}S^*(v)) = (u, SS^*(v))$ (since $S^{**} = S$) $= (u, v)$. Hence S^* is an isometry.

(iv) Let $P = [S]_B$. Then by 15.2 we have $[S^*]_B = \bar{P}^T$. Hence $I = [S^*S]_B = \bar{P}^T P$, and so P is unitary.

(v) The proof goes by induction on $n = \dim V$. The result is obvious for $n = 1$.

As $F = \mathbb{C}$, we can find a unit eigenvector v_1 of S. Let $S(v_1) = \lambda v_1$, and let $W = \text{Sp}(v_1)$. As $S^* = S^{-1}$, we have $S^*(v_1) = \lambda^{-1} v_1$.

We now show that $W^\perp$ is S-invariant: if $w \in W^\perp$, then

$$(S(w), v_1) = (w, S^*(v_1)) = (w, \lambda^{-1} v_1) = 0.$$

Hence $S(w) \in W^\perp$, and we have shows that $W^\perp$ is S-invariant. Now apply the induction hypothesis to the restriction $S_{W^\perp}$ (which is clearly an isometry of $W^\perp$): this gives us an orthonormal basis $v_2, \ldots, v_n$ of $W^\perp$ consistsing of S-evectors. Then $v_1, v_2, \ldots, v_n$ is an orthonormal basis of V consistsing of S-evectors. This completes the proof by induction.