Linear Algebra MATH 50003 Problem Sheet 9

1. (a) Let V be a finite-dimensional inner product space, let W be a subspace of V, and let $\pi_W : V \to W$ be the orthogonal projection map along W, as defined in lectures.

(i) For $v \in V$, show that $\pi_W(v)$ is the unique vector in W closest to v (ie. show $||v - w||$ $(w \in W)$ is minimal for $w = \pi_W(v)$).

(ii) If $v_1, \ldots, v_r$ is an orthonormal basis of W, prove that $\pi_W(v) = \sum_{i=1}^{r} (v, v_i)\, v_i$.

(b) (i) Let $V = \mathbb{R}^4$ with the usual inner product (ie. the dot product), and let $W = \mathrm{Sp}\left((1,1,0,0)^T,\ (1,1,1,2)^T\right)$. Find $w \in W$ such that $||w - (1,2,3,4)^T||$ is minimal.

(ii) Find a polynomial $p(x)$ over $\mathbb{R}$ of degree at most 3 such that $p(0) = p'(0) = 0$ and $\int_0^1 |2 + 3x - p(x)|^2\, dx$ is as small as possible.

2. Let V be a finite-dimensional inner product space, and let $S, T : V \to V$ be linear maps, with adjoints S^*, T^*. Prove the following statements:

(i) $(S+T)^* = S^* + T^*$

(ii) $(\lambda T)^* = \bar{\lambda} T^*$ for any scalar λ

(iii) $(T^*)^* = T$

(iv) $(ST)^* = T^* S^*$

(v) $\ker(T^*) = \mathrm{Im}(T)^\perp$

(vi) $\mathrm{Im}(T^*) = \ker(T)^\perp$.

3. Show that an $n \times n$ complex matrix P is unitary (ie. $P^T \bar{P} = I$) iff its columns form an orthonormal basis of $\mathbb{C}^n$. Show also that $|\lambda| = 1$ for any eigenvalue λ of P.

Find real numbers x, y, z such that the matrix $\frac{1}{2}\begin{pmatrix} 1 & x-i & z+i \\ yi & y & 1+i \\ 1+i & -1+i & x \end{pmatrix}$ is unitary.

Show that for any n, there are infinitely many diagonal complex unitary $n \times n$ matrices, but only finitely many diagonal real orthogonal $n \times n$ matrices.

4. For each of the following Hermitian matrices A, find a unitary matrix P such that $P^{-1}AP$ is diagonal:

$$\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ \frac{-i}{\sqrt{2}} & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 1 & i & -1+i \\ -i & 1 & -1-i \\ -1-i & -1+i & 0 \end{pmatrix}.$$

5. Let V be a finite-dimensional inner product space, and let $T : V \to V$ be a self-adjoint linear map.

(a) Show that $(T(v), v) \in \mathbb{R}$ for all $v \in V$.

(b) Define T to be *positive* if $(T(v), v) > 0$ for all nonzero $v \in V$. Prove that T is positive if and only if every eigenvalue of T is positive. (Recall that the eigenvalues of T are real, by 15.5 of lectures.)

(c) For the three Hermitian matrices A in Q4, the linear maps defined by $T(v) = Av$ for $v \in \mathbb{C}^n$ are self-adjoint. Which of these maps are positive?

(d) Show that if T is positive, then it has a self-adjoint positive *square root*, ie. a self-adjoint positive linear map S such that $S^2 = T$.

(e) Find a square root for each of the positive maps identified in part (c).

6. Let V be a finite-dimensional inner product space over F, where $F = \mathbb{R}$ or $\mathbb{C}$. We call a linear map $S : V \to V$ an *isometry* if $(S(u), S(v)) = (u, v)$ for all $u, v \in V$. For an isometry S, prove the following statements.

(i) $||S(v)|| = ||v||$ for all $v \in V$.

(ii) $S^* S = S S^* = I_V$.

(iii) The adjoint S^* is also an isometry.

(iv) If B is an orthonormal basis of V, then the matrix $[S]_B$ is unitary.

(v) Show that if $F = \mathbb{C}$, then there is an orthonormal basis E of V such that the matrix $[S]_E$ is diagonal. (Hint: try to copy the proof of the spectral theorem 15.3.)