Sheet 1 Solutions

1. (i) Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. Then $\mathbf{A}\cdot\mathbf{r} = xA_1 + yA_2 + zA_3$.
Thus $\nabla(\mathbf{A}\cdot\mathbf{r}) = \mathbf{i}\,\partial/\partial x(A_1x) + \mathbf{j}\,\partial/\partial y(A_2y) + \mathbf{k}\,\partial/\partial z(A_3z)$, since $\mathbf{A}$ is constant.
This simplifies to $\mathbf{i}\,A_1 + \mathbf{j}\,A_2 + \mathbf{k}\,A_3$. Thus $\nabla(\mathbf{A}\cdot\mathbf{r}) = \mathbf{A}$.
(ii) $\nabla(r^n) = \nabla(x^2+y^2+z^2)^{n/2} = (\mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z)(x^2+y^2+z^2)^{n/2}$
$= (n/2)(x^2+y^2+z^2)^{n/2-1}(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) = nr^{n-2}\mathbf{r}$.
(iii) $\mathbf{r}\cdot\nabla(x+y+z) = \mathbf{r}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = x+y+z$. Then $\nabla[\mathbf{r}\cdot\nabla(x+y+z)] = \nabla(x+y+z) = \mathbf{i}+\mathbf{j}+\mathbf{k}$.

2. $\nabla\phi = \mathbf{i}\,\partial/\partial x(x^2y+z^2x) + \mathbf{j}\,\partial/\partial y(x^2y+z^2x) + \mathbf{k}\,\partial/\partial z(x^2y+z^2x) = \mathbf{i}\,(2xy+z^2) + \mathbf{j}\,(x^2) + \mathbf{k}\,(2zx)$
$= 6\mathbf{i}+\mathbf{j}+4\mathbf{k}$ at the point $(1,1,2)$. A unit vector in the direction $(1,2,3)$ is $\widehat{\mathbf{s}} = (\mathbf{i}+2\mathbf{j}+3\mathbf{k})/\sqrt{(1+4+9)}$.
Then the directional derivative is $\widehat{\mathbf{s}}\cdot(\nabla\phi)_P = (1/\sqrt{14})(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\cdot(6\mathbf{i}+\mathbf{j}+4\mathbf{k}) = (1/\sqrt{14})(6+2+12) = 20/\sqrt{14}$.

3. Using the chain rule we have $d\phi/dt = (dx/dt)(\partial\phi/\partial x) + (dy/dt)(\partial\phi/\partial y) + (dz/dt)(\partial\phi/\partial z)$
$= (dx/dt, dy/dt, dz/dt)\cdot(\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z) = \mathbf{r}'(t)\cdot\nabla\phi$.
For ϕ as in Q2 we have $\phi = \cos^2 t\sin t + t^2\cos t$ and so $d\phi/dt = \cos^3 t - 2\cos t\sin^2 t + 2t\cos t - t^2\sin t$.
To check this is equal to $\mathbf{r}'(t)\cdot\nabla\phi$ we calculate $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$ and $\nabla\phi = (2xy+z^2, x^2, 2zx)$. Then $\mathbf{r}'(t)\cdot\nabla\phi = -(2xy+z^2)\sin t + x^2\cos t + 2zx$ which indeed equals $\cos^3 t - 2\cos t\sin^2 t + 2t\cos t - t^2\sin t$.
Similarly, if $\phi = \phi(g_1, g_2, g_3)$ then $d\phi/dt = (dg_1/dt)(\partial\phi/\partial g_1) + (dg_2/dt)(\partial\phi/\partial g_2) + (dg_3/dt)(\partial\phi/\partial g_3)$
$= \mathbf{g}'(t)\cdot\nabla_{\mathbf{g}}\phi$.

4. (i) Surface is given by $\phi = x^2+2y^2-z^2-8 = 0$. At $P\,(1,2,1)$ we have $(\nabla\phi)_P = 2\mathbf{i}+8\mathbf{j}-2\mathbf{k}$.
The equation of tangent plane is $(\mathbf{r}-\mathbf{r}_P)\cdot(\nabla\phi)_P = 0$, i.e. $((x-1)\mathbf{i}+(y-2)\mathbf{j}+(z-1)\mathbf{k})\cdot(2\mathbf{i}+8\mathbf{j}-2\mathbf{k}) = 0 \Rightarrow 2x-2+8y-16-2z+2 = 0 \Rightarrow x+4y-z = 8$.
(ii) This time we have $\phi = z - 3x^2y\sin(\pi x/2)$ and P is the point where $x = y = 1$ and therefore $z = 3\sin(\pi/2) = 3$. Thus $\nabla\phi = \mathbf{i}(-6xy\sin(\pi x/2) - 3x^2y(\pi/2)\cos(\pi x/2)) + \mathbf{j}(-3x^2\sin(\pi x/2)) + \mathbf{k}$, so that $(\nabla\phi)_P = -6\mathbf{i}-3\mathbf{j}+\mathbf{k}$. The equation of the tangent plane is therefore $((x-1)\mathbf{i}+(y-1)\mathbf{j}+(z-3)\mathbf{k})\cdot(-6\mathbf{i}-3\mathbf{j}+\mathbf{k}) = 0 \Rightarrow -6x+6-3y+3+z-3 = 0 \Rightarrow 6x+3y-z = 6$.

5. (i) $\nabla\phi = (\mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z)(x(x^2+y^2+z^2)) = \mathbf{i}\,(3x^2+y^2+z^2) + \mathbf{j}\,2xy + \mathbf{k}\,2zx$.
(ii) $\nabla\cdot(\phi\mathbf{r}) = \nabla\cdot(x^2r^2\,\mathbf{i} + xyr^2\,\mathbf{j} + xzr^2\,\mathbf{k}) = \partial/\partial x(x^2r^2) + x\partial/\partial y(yr^2) + x\partial/\partial z(zr^2) = 2xr^2 + xr^2 + xr^2 + x^2\partial r^2/\partial x + xy\partial r^2/\partial y + xz\partial r^2/\partial z$. Now $\partial/\partial x(r^2) = 2x, \partial/\partial y(r^2) = 2y$ and $\partial/\partial z(r^2) = 2z$.
Therefore we see that $\nabla\cdot(\phi\mathbf{r}) = 4xr^2 + 2x^3 + 2xy^2 + 2xz^2 = 6x^3 + 6xy^2 + 6xz^2 = 6xr^2$.
(iii) $\mathrm{curl}(f(r)\,\mathbf{r}) = \mathbf{i}\,(\partial/\partial y(zf) - \partial/\partial z(yf)) - \mathbf{j}\,(\partial/\partial x(zf) - \partial/\partial z(xf)) + \mathbf{k}\,(\partial/\partial x(yf) - \partial/\partial y(xf))$
$= \mathbf{i}\,(zf'(r)\partial r/\partial y - yf'(r)\partial r/\partial z) - \mathbf{j}\,(zf'(r)\partial r/\partial x - xf'(r)\partial r/\partial z) + \mathbf{k}\,(yf'(r)\partial r/\partial x - xf'(r)\partial r/\partial y)$.
Now $\partial r/\partial x = x/r, \partial r/\partial y = y/r, \partial r/\partial z = z/r$, so the above expression simplifies to
$f'(r)[\mathbf{i}(yz-yz)/r) - \mathbf{j}((xz-xz)/r) + \mathbf{k}((yx-xy)/r)] = \mathbf{0}$.

6. (i) $\mathbf{u}\times\mathbf{v} = (z^2,0,0)\times(x,y,z) = -z^3\,\mathbf{j} + z^2y\,\mathbf{k} \Rightarrow \nabla\cdot(\mathbf{u}\times\mathbf{v}) = 2zy$.
Now $\nabla\times\mathbf{u} = 2z\,\mathbf{j}$ and so $\mathbf{v}\cdot(\nabla\times\mathbf{u}) = 2yz$, while $\nabla\times\mathbf{v} = \mathbf{0}$ and so $\mathbf{u}\cdot(\nabla\times\mathbf{v})$ is also zero.
(ii) $\nabla\cdot(\psi\mathbf{u}) = \nabla\cdot(z^2(x^2+y^2+z^2)\mathbf{i}) = \partial/\partial x(z^2x^2) = 2z^2x$.
Now $\nabla\psi = (\mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z)(x^2+y^2+z^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and so $\nabla\psi\cdot\mathbf{u} = 2xz^2$,
while $\nabla\cdot\mathbf{u} = \partial/\partial x(z^2) = 0$ and so $\psi\nabla\cdot\mathbf{u} = 0$.

7. (i) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{b})_i(\mathbf{a}\times\mathbf{b})_i = \varepsilon_{ijk}a_jb_k\varepsilon_{ilm}a_lb_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(a_ja_lb_kb_m)$
$= a_ja_jb_kb_k - a_ja_kb_kb_j = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b})$.
(ii) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\times\mathbf{b})_i(\mathbf{c}\times\mathbf{d})_i = \varepsilon_{ijk}a_jb_k\varepsilon_{ilm}c_ld_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(a_jb_kc_ld_m)$
$= a_jc_jb_kd_k - a_jd_jb_kc_k = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$.
(iii) $[(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})]_i = \varepsilon_{ijk}(\mathbf{a}\times\mathbf{b})_j(\mathbf{c}\times\mathbf{d})_k = \varepsilon_{ijk}\varepsilon_{jlm}a_lb_m\varepsilon_{kpq}c_pd_q = \varepsilon_{kij}\varepsilon_{kpq}a_lb_m\varepsilon_{jlm}c_pd_q$
$= \varepsilon_{jlm}(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})(a_lb_mc_pd_q) = \varepsilon_{jlm}a_lb_m(c_id_j - d_ic_j)$
$= c_i(\mathbf{a}\times\mathbf{b})_jd_j - d_i(\mathbf{a}\times\mathbf{b})_jc_j = [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]c_i - [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]d_i$.

8. (i) $\delta_{ij}\partial x_i/\partial x_j = \partial x_i/\partial x_i = 1+1+1 = 3$.
(ii) $\delta_{ij}\delta_{ik}x_jx_k = x_ix_i = x_1^2 + x_2^2 + x_3^2 = |\mathbf{r}|^2$.
(iii) $\delta_{ij}\partial^2\phi/\partial x_i\partial x_j = \partial^2\phi/\partial x_i^2 = \partial^2\phi/\partial x_1^2 + \partial^2\phi/\partial x_2^2 + \partial^2\phi/\partial x_3^2 = \nabla^2\phi$.
(iv) $\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ij}(\delta_{jk}\delta_{ki}) = \delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$.
(v) $\varepsilon_{ijk}\partial/\partial x_i(\partial A_k/\partial x_j) = (1/2)\varepsilon_{ijk}\partial/\partial x_i(\partial A_k/\partial x_j) + (1/2)\varepsilon_{jik}\partial/\partial x_j(\partial A_k/\partial x_i)$
$= (1/2)\varepsilon_{ijk}(\partial/\partial x_i(\partial A_k/\partial x_j) - \partial/\partial x_j(\partial A_k/\partial x_i)) = 0$.

9. (i) $[\nabla \times (\phi \mathbf{A})]_i = \varepsilon_{ijk}\partial(\phi A_k)/\partial x_j = \phi\varepsilon_{ijk}\partial A_k/\partial x_j + \varepsilon_{ijk}\partial\phi/\partial x_j A_k = \phi[\nabla \times \mathbf{A}]_i + \varepsilon_{ijk}(\nabla\phi)_j A_k$
$= \phi[\nabla \times \mathbf{A}]_i + [\nabla\phi \times \mathbf{A}]_i$.

(ii) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \partial/\partial x_i(\mathbf{A} \times \mathbf{B})_i = (\partial/\partial x_i)(\varepsilon_{ijk}A_j B_k) = \varepsilon_{ijk}\,(B_k\partial A_j/\partial x_i + A_j\partial B_k/\partial x_i)$
$= \varepsilon_{kij}(\partial A_j/\partial x_i)B_k - \varepsilon_{jik}A_j\partial B_k/\partial x_i = (\nabla \times \mathbf{A})_k B_k - A_j(\nabla \times \mathbf{B})_j = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$.

(iii) $[\mathbf{A} \times (\nabla \times \mathbf{A})]_i = \varepsilon_{ijk}A_j(\nabla \times \mathbf{A})_k = \varepsilon_{ijk}A_j\varepsilon_{klm}\partial A_m/\partial x_l = \varepsilon_{kij}\varepsilon_{klm}A_j\partial A_m/\partial x_l$
$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(A_j\partial A_m/\partial x_l) = A_j\partial A_j/\partial x_i - A_j\partial A_i/\partial x_j = (1/2)\partial A_j^2/\partial x_i - [\mathbf{A} \cdot \nabla]A_i$
$= (1/2)[\nabla(\mathbf{A} \cdot \mathbf{A})]_i - [(\mathbf{A} \cdot \nabla)\mathbf{A}]_i$.