Problem Sheet 1

1. If $\mathbf{A}$ is a constant vector field, calculate the gradients of the following scalar fields:

$$\begin{array}{ll} \text{(i)} & \mathbf{A}\cdot\mathbf{r}, \\ \text{(ii)} & r^n, \\ \text{(iii)} & \mathbf{r}\cdot\nabla(x+y+z), \end{array}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$.

2. If $\phi = x^2y + z^2x$ and P is the point $(1, 1, 2)$, find the directional derivative of ϕ at P in the direction $(1, 2, 3)$.

3. If $\phi = \phi(\mathbf{r})$ with $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $x = x(t), y = y(t), z = z(t)$, show that

$$\frac{d\phi}{dt} = \mathbf{r}'(t)\cdot\nabla\phi.$$

Verify this relation for ϕ as given in Q2 and $(x, y, z) = (\cos t, \sin t, t)$.
Further, if $\phi = \phi(\mathbf{g}(t))$ with $\mathbf{g} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$, show that

$$\frac{d\phi}{dt} = \mathbf{g}'(t)\cdot\nabla_{\mathbf{g}}\phi.$$

where $\nabla_{\mathbf{g}} \equiv \mathbf{i}\frac{\partial}{\partial g_1} + \mathbf{j}\frac{\partial}{\partial g_2} + \mathbf{k}\frac{\partial}{\partial g_3}$.

4. Find the equations of the tangent planes to the following surfaces at the points indicated

$$\begin{array}{ll} \text{(i)} & x^2 + 2y^2 - z^2 - 8 = 0 \text{ at } (1, 2, 1), \\ \text{(ii)} & z = 3x^2y\sin(\pi x/2) \text{ at } x = 1, y = 1. \end{array}$$

5. If $\phi = xr^2, \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(r)$ is an arbitrary function of $r = |\mathbf{r}|$, evaluate:

$$\begin{array}{ll} \text{(i)} & \nabla\phi, \\ \text{(ii)} & \operatorname{div}(\phi\mathbf{r}), \\ \text{(iii)} & \operatorname{curl}(f(r)\mathbf{r}). \end{array}$$

6. If $\mathbf{u} = z^2\mathbf{i}, \mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\psi = |\mathbf{v}|^2$, verify the identities

$$\begin{array}{ll} \text{(i)} & \operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v}, \\ \text{(ii)} & \operatorname{div}(\psi\mathbf{u}) = (\nabla\psi)\cdot\mathbf{u} + \psi\operatorname{div}\mathbf{u}. \end{array}$$

7. Use tensor notation and the relation $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ to establish the following vector identities:

$$\begin{array}{ll} \text{(i)} & (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2, \\ \text{(ii)} & (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}), \\ \text{(iii)} & (\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = ((\mathbf{a}\times\mathbf{b})\cdot\mathbf{d})\mathbf{c} - ((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\mathbf{d}, \end{array}$$

8. Simplify the following expressions:

$$\begin{array}{ll} \text{(i)} & \delta_{ij}\partial x_i/\partial x_j, \\ \text{(ii)} & \delta_{ij}\delta_{ik}x_jx_k, \\ \text{(iii)} & \delta_{ij}\partial^2\phi/\partial x_i\partial x_j, \\ \text{(iv)} & \delta_{ij}\delta_{jk}\delta_{ki}, \\ \text{(v)} & \varepsilon_{ijk}\partial/\partial x_i\left(\partial A_k/\partial x_j\right). \end{array}$$

9. Use tensor notation to prove the following identities:

$$\begin{array}{ll} \text{(i)} & \operatorname{curl}(\phi\mathbf{A}) = \phi\,\operatorname{curl}\mathbf{A} + \nabla\phi\times\mathbf{A}, \\ \text{(ii)} & \operatorname{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\operatorname{curl}\mathbf{A} - \mathbf{A}\cdot\operatorname{curl}\mathbf{B}, \\ \text{(iii)} & \mathbf{A}\times\operatorname{curl}\mathbf{A} = \frac{1}{2}\nabla(|\mathbf{A}|^2) - (\mathbf{A}\cdot\nabla)\mathbf{A}. \end{array}$$

Sheet 1 Answers

1. (i) $\mathbf{A}$; (ii) $nr^{n-2}\mathbf{r}$; (iii) $\mathbf{i}+\mathbf{j}+\mathbf{k}$.
2. $20/\sqrt{14}$.
3. $d\phi/dt = \cos^3 t - 2\cos t\sin^2 t + 2t\cos t - t^2\sin t$.
4. (i) $x + 4y - z = 8$; (ii) $6x + 3y - z = 6$.
5. (i) $(3x^2 + y^2 + z^2)\mathbf{i} + 2xy\mathbf{j} + 2zx\mathbf{k}$; (ii) $6xr^2$; (iii) zero.
8. (i) 3; (ii) r^2; (iii) $\partial^2\phi/\partial x_1^2 + \partial^2\phi/\partial x_2^2 + \partial^2\phi/\partial x_3^2$; (iv) 3; (v) zero.