Sheet 2 Solutions

1. We have that

$$\begin{aligned}\text{curl}\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^2 & 2yz+x^2 & 2xz+y^2 \end{vmatrix} \\ &= \mathbf{i}(2y-2y) - \mathbf{j}(2z-2z) + \mathbf{k}(2x-2x) \\ &= \mathbf{0}.\end{aligned}$$

If $\mathbf{v} = \nabla\phi$, we have

$$\partial\phi/\partial x = 2xy+z^2, \ \ \partial\phi/\partial y = 2yz+x^2, \ \ \partial\phi/\partial z = 2xz+y^2.$$

Integrating the first equation we have $\phi = yx^2+z^2x+f(y,z)$, while the second gives $\phi = zy^2+yx^2+g(x,z)$ and the third gives $\phi = xz^2+y^2z+h(x,y)$. Putting all this together, we deduce that

$$\phi(x,y,z) = yx^2+xz^2+zy^2+C,$$

where C is zero since $\phi = 0$ at $(x,y,z) = (0,0,0)$. It follows that

$$\int_P \mathbf{v}\cdot d\mathbf{r} = [\phi]_{(0,0,0)}^{(1,2,3)} = \phi(1,2,3) = 2+9+12 = 23.$$

2. The straight line from $A(0,0,0)$ to $B(1,2,3)$ can be parametrized by $x = t, y = 2t, z = 3t$ $(0 \le t \le 1)$. Then

$$\begin{aligned} I &= \int_0^1 (2t^2 dt + 6t^2 2\,dt + 3t^2 3\,dt) \\ &= \int_0^1 23t^2\,dt = 23/3.\end{aligned}$$

3. Firstly, we compute the quantity

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r} &= (3x^2\mathbf{i} + (2xz-y)\mathbf{j} + z\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}) \\ &= 3x^2dx + (2xz-y)dy + zdz.\end{aligned}$$

(i) The straight line can be written as $(x,y,z) = (2u,u,3u)$ for $0 \le u \le 1$. The integral is then

$$\begin{aligned} I_1 &= \int_0^1 3(4u^2)(2\,du) + (2(2u)(3u)-u)\,du + (3u)\,3\,du \\ &= \int_0^1 (36u^2+8u)\,du = 16.\end{aligned}$$

(ii) Substituting in the given parametrization:

$$\begin{aligned} I_2 &= \int_0^1 3(2t^2)^2.4t\,dt + (2(2t^2)(4t^2-t)-t).dt + (4t^2-t)(8t-1)\,dt \\ &= \int_0^1 (48t^5+16t^4-4t^3-t+32t^3-12t^2+t)\,dt \\ &= 8+16/5-1+8-4 = 71/5.\end{aligned}$$

(iii) Following the same method:

$$\begin{aligned} I_3 &= \int_0^2 3s^2\,ds + ((2s)(3s^3/8)-s^2/4)\,(s/2)\,ds + (3s^3/8)\,(9s^2/8)\,ds \\ &= \int_0^2 3s^2+3s^5/8-s^3/8+27s^5/64\,ds \\ &= 8+4-1/2+9/2 = 16.\end{aligned}$$

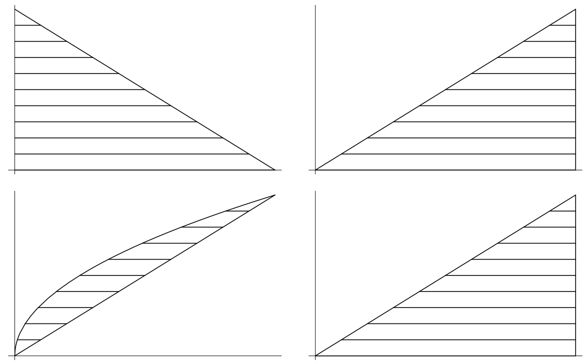

Figure 1: The regions of integration for the integrals in Q4.

4. The regions of integration for the 4 integrals are given above.

(i) (b) $I = \int_0^a (a-x)\,dx = a^2/2$; (c) & (d) $\int_0^a (\int_0^{a-y} dx)\,dy = \int_0^a (a-y)\,dy = a^2/2$;

(ii) (b) $I = \int_0^a \left[x^2y + \frac{1}{3}y^3\right]_0^x dx = \int_0^a \frac{4}{3}x^3\,dx = a^4/3$;

(ii) (c) & (d) $\int_0^a \left(\int_y^a (x^2+y^2)\,dx\right) dy = \int_0^a \left[\frac{1}{3}x^3 + y^2x\right]_y^a dy = \int_0^a \frac{1}{3}a^3 + ay^2 - \frac{1}{3}y^3 - y^3\,dy = a^4/3.$

(iii) (b) $I = \int_0^1 \left[xy^3/3\right]_x^{\sqrt{x}} dx = \frac{1}{3}\int_0^1 (x^{5/2} - x^4)\,dx = \frac{1}{3}(\frac{2}{7} - \frac{1}{5}) = 1/35.$

(iii) (c) & (d) $\int_0^1 \left(\int_{y^2}^y xy^2 dx\right) dy = -\int_0^1 \left[y^2x^2/2\right]_y^{y^2} dy = -\frac{1}{2}\int_0^1 (y^6 - y^4)\,dy = 1/35.$

(iv) (b) $I = \int_0^1 xe^{-x^2}dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1 = \frac{1}{2}(1-e^{-1}).$

(iv) (c) & (d) Changing the order of integration we get $\int_0^1 \left(\int_y^1 e^{-x^2}dx\right) dy$. The inner integral cannot be evaluated.

5. First we need to calculate the normal to the cylinder. The surface is described by $\phi = y^2 - 8x = 0$ with $0 \le z \le 6$. A unit normal is therefore

$$\frac{\nabla\phi}{|\nabla\phi|} = \pm\frac{-8\mathbf{i} + 2y\mathbf{j}}{\sqrt{64+4y^2}} = \pm\frac{y\mathbf{j} - 4\mathbf{i}}{\sqrt{16+y^2}}.$$

The unit normal pointing in the direction of increasing x is therefore

$$\widehat{\mathbf{n}} = \frac{4\mathbf{i} - y\mathbf{j}}{\sqrt{16+y^2}},$$

and so

$$\mathbf{F}\cdot\widehat{\mathbf{n}} = \frac{8y + yz}{\sqrt{16+y^2}}.$$

The required integral is therefore

$$I = \int_S \frac{8y+yz}{\sqrt{16+y^2}}\,dS = \int_\Sigma \frac{8y+yz}{\sqrt{16+y^2}}\frac{dy\,dz}{|\widehat{\mathbf{n}}\cdot\mathbf{i}|}$$

where we have used the projection theorem to project onto the plane $x = 0$. The projected shape will be a rectangle with $0 \le y \le 4$ and $0 \le z \le 6$. We therefore have

$$\begin{aligned} I &= \frac{1}{4}\int_0^6 \int_0^4 (8y + yz)\,dy\,dz \\ &= \frac{1}{4}\int_0^6 [4y^2 + zy^2/2]_0^4\,dz \\ &= \frac{1}{4}\int_0^6 (64 + 8z)\,dz \\ &= (16)(6) + 36 = 132. \end{aligned}$$

6. Firstly we have that

$$\begin{aligned}\text{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-2xz & -xy \end{vmatrix} \\ &= \mathbf{i}(-x+2x) - \mathbf{j}(-y) + \mathbf{k}(1-2z-1) \\ &= x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}.\end{aligned}$$

Next we need to find the normal to the hemisperical surface. The surface is described by $\phi = x^2+y^2+z^2 = a^2$ and hence a unit normal is

$$\frac{\nabla\phi}{|\nabla\phi|} = \pm\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{4x^2+4y^2+4z^2}} = \pm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}.$$

Since we want the outward normal we choose the positive sign so that

$$\widehat{\mathbf{n}} = (1/a)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

The required integral is therefore

$$I = \frac{1}{a}\int_S (x^2+y^2-2z^2)\,dS = \frac{1}{a}\int_S (3x^2+3y^2-2a^2)\,dS,$$

upon substituting for z. We now project this down onto the plane $z=0$ using the projection theorem. The projected surface Σ will be a cricle of radius a. The integral becomes

$$\begin{aligned} I &= \frac{1}{a}\int_\Sigma (3x^2+3y^2-2a^2)\frac{d\Sigma}{|\widehat{\mathbf{n}}\cdot\mathbf{k}|} \\ &= \int_\Sigma (3x^2+3y^2-2a^2)\frac{d\Sigma}{z} \\ &= \int_\Sigma (3x^2+3y^2-2a^2)(a^2-x^2-y^2)^{-1/2}d\Sigma.\end{aligned}$$

As suggested in the hint, the best way to deal with this double integral, bearing in mind that Σ is a circle of radius a, is to transform to plane polar coordinates by writing $x = r\cos\theta, y = r\sin\theta, d\Sigma = dxdy = rdrd\theta$. The integral becomes

$$\begin{aligned} & \int_0^{2\pi}\int_0^a (3r^2-2a^2)(a^2-r^2)^{-1/2}r\,dr\,d\theta \\ =\ & 2\pi\int_0^a \left[-3(a^2-r^2)+a^2\right](a^2-r^2)^{-1/2}r\,dr \\ =\ & 2\pi\int_0^a -3r(a^2-r^2)^{1/2}+a^2r(a^2-r^2)^{-1/2}\,dr \\ =\ & 2\pi\left[(a^2-r^2)^{3/2}-a^2(a^2-r^2)^{1/2}\right]_0^a \\ =\ & 2\pi(-a^3+a^3) = 0.\end{aligned}$$

After all that work, the answer was zero! Later we will see that using either the divergence theorem or Stokes theorem this result can be obtained much more easily.

7. First we note that $x^2+y^2 = a^2$ so that the surface in question is the curved surface of half of a cylinder of radius a and height $(H_2 - H_1)$. (The 'half' arises because θ is restricted to $[0,\pi]$).
The unit normal to S is given by

$$\widehat{\mathbf{n}} = \nabla(x^2+y^2)/\left|\nabla(x^2+y^2)\right| = \pm(2x\mathbf{i}+2y\mathbf{j})/(4x^2+4y^2)^{1/2} = \pm(x\mathbf{i}+y\mathbf{j})/a.$$

Projecting onto $y=0$ we have

$$dS = dx\,dz/\left|\widehat{\mathbf{n}}\cdot\mathbf{j}\right| = dx\,dz/(y/a) = a\,dx\,dz/(a^2-x^2)^{1/2}.$$

The projected region Σ has $x = a\cos\theta, y = 0, z = \lambda$ for $H_1 \le \lambda \le H_2$, $0 \le \theta \le \pi$ and hence is the rectangular region with $-a \le x \le a, H_1 \le z \le H_2$. The surface area is therefore

$$\int_{H_1}^{H_2}\int_{-a}^{a}\frac{a}{(a^2-x^2)^{1/2}}\,dx\,dz = a(H_2-H_1)\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_{-a}^{a} = \pi a(H_2-H_1).$$

8. Calculating the double integral first we have

$$\begin{aligned}\int_0^b\left(\int_0^a (2y-a)\,dx\right)dy &= \int_0^b a(2y-a)\,dy\\ &= a\left[y^2-ay\right]_0^b\\ & ab^2-a^2b.\end{aligned}$$

Now we need to evaluate the line integral. We split it up into the four parts around the sides of the rectangle. If we start on the x axis then we have $x = t, y = 0$ for $0 \le t \le a$. Call this path P_1. On this path $L = 0$ and $M = 0$. Then

$$\int_{P_1} = 0.$$

Next we travel from $(a, 0)$ to (a, b). Along this path (P_2) we have $x = a$ and $y = t$ for $0 \le t \le b$, and hence $dx = 0, dy = dt$. On this path $L = at, M = 2at$. Then

$$\int_{P_2} = \int_0^b at(0) + 2at\,dt = ab^2.$$

Remembering that we are travelling in an anti-clockwise fashion we now travel along P_3 from (a, b) to $(0, b)$ so that $x = t, y = b$ with t starting at a and finishing at 0. Along P_3 we have $dx = dt$, but $dy = 0$. We also have $L = ab$ and $M = 2bt$. The contribution to the integral is

$$\int_{P_3} = \int_a^0 ab\;dt = -a^2b.$$

Finally we travel back from $(0, b)$ to $(0, 0)$ so we have $x = dx = 0$ and $y = t$ with t starting at b and finishing at 0. The functions are $L = at, M = 0$. The integral is

$$\int_{P_4} = \int_b^0 at(0) + (0)\,dt = 0.$$

Adding all four contributions we calculate that

$$\begin{aligned}\oint_C &= \int_{P_1} + \int_{P_2} + \int_{P_3} + \int_{P_4}\\ &= ab^2 - a^2b.\end{aligned}$$

This value agrees with that found from the double integral and therefore Green's theorem is verified.

9. Taking Green's theorem with $L = -y$ and $M = x$ we have $\partial M/\partial x - \partial L/\partial y = 2$ and so

$$\oint_C x\,dy - y\,dx = \int_R 2\,dx\,dy = 2 \times \text{area of } R.$$

Hence the area is equal to $(1/2)(\oint_C x\,dy - y\,dx)$ as required.

Turning now to the cycloid, we have to split the curve C into the part (C_1 say) along the x-axis from $x = 0$ to $x = 2\pi a$ with $y = 0$ and then there is the contribution along the arc of the cycloid (C_2) with t starting at 2π and ending at zero.

$$\int_{C_1}(xdy - ydx) = 0,$$

since y and dy are both zero. Along C_2 :

$$\begin{aligned}
\int_{C_2} (xdy - ydx) &= \int_{2\pi}^{0} a(t - \sin t)\, a \sin t \, dt - a(1 - \cos t)\, a(1 - \cos t)\, dt \\
&= a^2 \int_{2\pi}^{0} t \sin t - \sin^2 t - 1 - \cos^2 t + 2 \cos t \, dt \\
&= a^2 \left\{ [-t \cos t + 2 \sin t - 2t]_{2\pi}^{0} + \int_{2\pi}^{0} \cos t \, dt \right\} \\
&= a^2 (2\pi + 4\pi) = 6\pi a^2.
\end{aligned}$$

Therefore the required area is one half of this, namely $3\pi a^2$.