Problem Sheet 2

1. The vector field $\mathbf{v}$ is given by

$$\mathbf{v} = (2xy + z^2)\,\mathbf{i} + (2yz + x^2)\,\mathbf{j} + (2xz + y^2)\,\mathbf{k}.$$

Show that $\operatorname{curl}\mathbf{v} = 0$ and find the potential ϕ such that $\mathbf{v} = \nabla\phi$ with $\phi = 0$ at the origin. Hence evaluate the line integral

$$\int_P \mathbf{v}\cdot d\mathbf{r},$$

where P is any path joining $(0,0,0)$ to $(1,2,3)$.

2. Evaluate the line integral

$$I = \int_P (xy\,dx + yz\,dy + zx\,dz)$$

where P is the straight line joining the starting point $A(0,0,0)$ and the end point $B(1,2,3)$.

3. The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = 3x^2\,\mathbf{i} + (2xz - y)\,\mathbf{j} + z\,\mathbf{k}.$$

Evaluate the line integral

$$\int_P \mathbf{F}\cdot d\mathbf{r}$$

along each of the following paths between $(0,0,0)$ and $(2,1,3)$:

(i) a straight line;
(ii) the curve defined by $(2t^2, t, 4t^2 - t)$ with $0 \le t \le 1$;
(iii) the curve defined by $(s, s^2/4, 3s^3/8)$ with $0 \le s \le 2$.

4. In each of the following cases (a) sketch the region of integration; (b) evaluate the integral; (c) write down the integral with the order of integration reversed; (d) evaluate the integral again (if possible) and compare with (b):

(i) $\int_0^a \left(\int_0^{a-x} dy\right) dx$; (ii) $\int_0^a \left(\int_0^x (x^2+y^2)\,dy\right) dx$;
(iii) $\int_0^1 \left(\int_x^{\sqrt{x}} xy^2\,dy\right) dx$ (iv) $\int_0^1 \left(\int_0^x e^{-x^2}\,dy\right) dx$.

5. If

$$\mathbf{F} = 2y\,\mathbf{i} - z\,\mathbf{j} + x^2\,\mathbf{k},$$

and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant, bounded by the planes $y = 4$ and $z = 6$, evaluate

$$\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS,$$

where $\widehat{\mathbf{n}}$ points in the direction of increasing x, by projecting the integral onto the plane $x = 0$.

6. If

$$\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$$

and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the $x - y$ plane, evaluate

$$\int_S (\nabla\times\mathbf{F})\cdot\widehat{\mathbf{n}}\,dS,$$

where $\widehat{\mathbf{n}}$ is the unit normal out of the sphere. (Hint: project onto the $x - y$ plane and use plane polar coordinates (r,θ) to evaluate the resulting integral - you may assume that $dxdy = rdrd\theta$).

7. The open surface S is described parametrically by

$$x = a\cos\theta,\ \ y = a\sin\theta,\ \ z = \lambda,\ \ (H_1 \le \lambda \le H_2,\ 0 \le \theta \le \pi),$$

where a, H_1, H_2 are constants. Calculate the surface area of S by projecting the surface onto the plane $y = 0$.

8. Verify Green's theorem in the plane

$$\oint_C (L\,dx + M\,dy) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

for the special case where C is the boundary of the rectangle with vertices $(0,0), (a,0), (a,b), (0,b)$ and $L = ay, M = 2xy$.

9. Use Green's theorem to show that the area enclosed by a simple closed curve with boundary C can be expressed as

$$\frac{1}{2}\oint_C (x\,dy - y\,dx).$$

Use this result to calculate the area bounded by one arc of the cycloid

$$x = a(t - \sin t),\ \ y = a(1 - \cos t),$$

(with $a > 0$, and $0 \le t \le 2\pi$) and the $x-$axis.

Sheet 2 Answers

1. $\phi = yx^2 + xz^2 + zy^2$; value of integral is 23.
2. 23/3.
3. (i) 16; (ii) 71/5; (iii) 16.
4. (i) $a^2/2$; (ii) $a^4/3$; (iii) 1/35; (iv) $\frac{1}{2}(1 - \mathrm{e}^{-1})$.
5. 132.
6. zero.
7. $\pi a(H_2 - H_1)$.
8. LHS = RHS = $ab^2 - a^2b$.
9. $3\pi a^2$.