Problem Sheet 3

1. A region V is enclosed by a surface S. ϕ and $\mathbf{A}$ are scalar and vector fields with ϕ vanishing on S. Show, by applying the divergence theorem to $\phi\mathbf{A}$, that

$$\int_V \phi \,\mathrm{div}\,\mathbf{A}\, dV = -\int_V \mathbf{A}\cdot\nabla\phi\, dV.$$

Deduce that if $\mathbf{A}$ is solenoidal throughout V then

$$\int_V \mathbf{A}\cdot\nabla\phi\, dV = 0.$$

Show that in two dimensions the corresponding result is

$$\int_R \phi \,\mathrm{div}\,\mathbf{A}\, dx\, dy = -\int_R \mathbf{A}\cdot\nabla\phi\, dx\, dy,$$

where ϕ vanishes on the closed curve C which bounds the region R.

2. Evaluate

$$\int_S \mathbf{r}\cdot\widehat{\mathbf{n}}\, dS$$

where S is any closed surface enclosing a volume V, and $\mathbf{r}$ is the position vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

3. Show that

$$\int_S \frac{\mathbf{r}\cdot\widehat{\mathbf{n}}}{r^2}\, dS = \int_V \frac{dV}{r^2},$$

where S is a closed surface enclosing the volume V, and $\mathbf{r}$ is as above, with $r = |\mathbf{r}|$.

4. Use the divergence theorem to prove the following results, where S is a closed surface with unit outward normal $\widehat{\mathbf{n}}$ enclosing the volume τ, $\phi(x,y,z)$ is a scalar and $\mathbf{A}(x,y,z)$ a vector function of position.

$$\text{(i)}\quad \int_S \widehat{\mathbf{n}}\,\phi\, dS = \int_\tau \nabla\phi\, d\tau. \quad \text{(ii)}\quad \int_S \widehat{\mathbf{n}}\times\mathbf{A}\, dS = \int_\tau \mathrm{curl}\,\mathbf{A}\, d\tau.$$

5. Verify the divergence theorem for the case when

$$\mathbf{A} = x\,\mathbf{i}$$

and V is the cube $|x|\le a$, $|y|\le a$, $|z|\le a$.

6. Let S be the piecewise smooth closed surface consisting of the surface of the cone $z=(x^2+y^2)^{1/2}$ for $x^2+y^2\le 1$, together with the flat cap consisting of the disk $x^2+y^2\le 1$ in the plane $z=1$. Verify the divergence theorem

$$\int_S \mathbf{A}\cdot\widehat{\mathbf{n}}\; dS = \int_V \mathrm{div}\,\mathbf{A}\, dV$$

for this surface, when $\mathbf{A}=(x+y)\mathbf{i}+(y-x-z)\mathbf{j}+(z-y)\mathbf{k}$. You may assume $dx\,dy = r\,dr\,d\theta$ in plane polar coordinates.

7. By converting into an appropriate line integral, use Stokes theorem to evaluate

$$\int_S (\nabla\times\mathbf{A})\cdot\widehat{\mathbf{n}}\, dS$$

where $\mathbf{A}=(y-z, z-x, x-y)$. Here S is the upper half of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ with $z\ge 0$, and $\widehat{\mathbf{n}}$ is the unit normal to S with $\widehat{\mathbf{n}}\cdot\mathbf{k}>0$.

8. Verify Stokes theorem for the vector field

$$\mathbf{A} = (3x - y, -yz^2/2, -y^2 z/2)$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = a^2$, so that the closed curve C is a circle in the $x - y$ plane.

Hint: to evaluate the surface integral use spherical polar coordinates $x = a \sin\theta \cos\phi, y = a \sin\theta \sin\phi, z = a \cos\theta$, with $dS = a^2 \sin\theta \, d\theta \, d\phi$, and $0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$.

9. Let S consist of the part of the cone $z = (x^2 + y^2)^{1/2}$ for $x^2 + y^2 \le 9$ and suppose

$$\mathbf{A} = (-y, x, -xyz).$$

Verify that Stokes theorem is satisfied for this choice of $\mathbf{A}$ and S.

Sheet 3 Answers

2. $3V$.
5. LHS = RHS = $8a^3$.
6. LHS = RHS = π.
7. $-2\pi ab$.
8. LHS = RHS = πa^2.
9. LHS = RHS = 18π.