Sheet 4 Solutions

1. $(u,v)=(1,0)\Rightarrow(x,y)=(1,1)$. Consider

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{pmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{pmatrix}=\begin{pmatrix}3u^2+v & u+3v^2\\ 2u & -2v\end{pmatrix}.$$

Then

$$\det(J)=-2v(3u^2+v)-2u(u+3v^2)=-2$$

when $(u,v)=(1,0)$. Since $\det(J)\neq 0$ at this point, the inverse function theorem tells us that locally the expressions for x and y can be inverted.

To find u_x and v_x, differentiate the expressions for x and y implicitly with respect to x to get

$$1=3u^2u_x+vu_x+uv_x+3v^2v_x,\quad 0=2uu_x-2vv_x.$$

This can be rewritten as

$$\begin{pmatrix}3u^2+v & u+3v^2\\ 2u & -2v\end{pmatrix}\begin{pmatrix}u_x\\ v_x\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}.$$

(Note that the matrix on the left is the Jacobian calculated earlier). Thus, after substituting $(u,v)=(1,0)$:

$$\begin{pmatrix}u_x\\ v_x\end{pmatrix}=\begin{pmatrix}3 & 1\\ 2 & 0\end{pmatrix}^{-1}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}0 & 1/2\\ 1 & -3/2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Similarly, by differentiating with respect to y and setting $(u,v)=(1,0)$:

$$\begin{pmatrix}3 & 1\\ 2 & 0\end{pmatrix}\begin{pmatrix}u_y\\ v_y\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix},$$

from which we obtain

$$\begin{pmatrix}u_y\\ v_y\end{pmatrix}=\begin{pmatrix}3 & 1\\ 2 & 0\end{pmatrix}^{-1}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}0 & 1/2\\ 1 & -3/2\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}1/2\\ -3/2\end{pmatrix}.$$

2. (i) First we differentiate our expressions implicitly with respect to x. This gives

$$2uu_x+4vv_x=2x,\quad vyu_x+u\,yv_x-xyv_x=vy,$$

which can be written in matrix form as

$$\begin{pmatrix}2u & 4v\\ vy & uy-xy\end{pmatrix}\begin{pmatrix}u_x\\ v_x\end{pmatrix}=\begin{pmatrix}2x\\ vy\end{pmatrix}.$$

Differentiating the original expressions with respect to y :

$$2uu_y+4vv_y=-2y,\quad vyu_y+u\,yv_y-xyv_y=vx-uv,$$

i.e.

$$\begin{pmatrix}2u & 4v\\ vy & uy-xy\end{pmatrix}\begin{pmatrix}u_y\\ v_y\end{pmatrix}=\begin{pmatrix}-2y\\ vx-uv\end{pmatrix}.$$

In both cases the derivatives exist at (x_0,y_0,u_0,v_0) provided

$$\det(J)=\det\begin{pmatrix}2u_0 & 4v_0\\ v_0y_0 & u_0y_0-x_0y_0\end{pmatrix}\neq 0$$

as required.

Consider $(x,y,u,v)=(1,1,2,1)$. Then $u^2+2v^2+y^2-x^2=6$ and $uvy-vxy=1$ so that both equations are satisfied. In this case

$$\det(J)=\det\begin{pmatrix}4 & 4\\ 1 & 1\end{pmatrix}=0$$

and so the partial derivatives are undefined at this point.

Now let $(x, y, u, v) = (1+\sqrt{2}, -1-\sqrt{2}, 2, 1)$. Then $u^2+2v^2+y^2-x^2 = 2^2+2+(1+\sqrt{2})^2-(1+\sqrt{2})^2 = 6$ and $uvy - vxy = -2(1+\sqrt{2}) + (1+\sqrt{2})^2 = 1$, so that again both equations are satisfied. This time

$$\det(J) = \det \begin{pmatrix} 4 & 4 \\ -1-\sqrt{2} & 1 \end{pmatrix} \neq 0$$

and so at this point the partial derivatives u_x, v_x, u_y, v_y do exist.

(ii) Here $F_1(x, y, u, v) = u^2+2v^2+y^2-x^2-6$ and $F_2(x, y, u, v) = uvy-vxy-1$, and so (with x, y, u, v treated as independent quantities):

$$\begin{pmatrix} \partial F_1/\partial u & \partial F_1/\partial v \\ \partial F_2/\partial u & \partial F_2/\partial v \end{pmatrix} = \begin{pmatrix} 2u & 4v \\ vy & uy-xy \end{pmatrix},$$

and so the determinant of this matrix at (x_0, y_0, u_0, v_0) is indeed the determinant derived in part (i).

(iii) We saw in part (i) that two distinct points (x_0, y_0) map to the same (u_0, v_0). This would mean for example that two particles occupying different locations in the $x-y$ plane would occupy the same location in the $u-v$ plane.

3. To show that the system is orthogonal we have to work out $\partial \mathbf{r}/\partial \xi$ and $\partial \mathbf{r}/\partial \eta$ and show that these vectors are orthogonal.

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \xi} &= \frac{\partial x}{\partial \xi}\mathbf{i} + \frac{\partial y}{\partial \xi}\mathbf{j} \\ &= \left(\frac{c\cosh\xi}{\cosh\xi-\cos\eta} - \frac{c\sinh^2\xi}{(\cosh\xi-\cos\eta)^2}\right)\mathbf{i} - \frac{c\sin\eta\sinh\xi}{(\cosh\xi-\cos\eta)^2}\mathbf{j}. \\ \frac{\partial \mathbf{r}}{\partial \eta} &= \frac{\partial x}{\partial \eta}\mathbf{i} + \frac{\partial y}{\partial \eta}\mathbf{j} \\ &= \frac{-c\sin\eta\sinh\xi}{(\cosh\xi-\cos\eta)^2}\mathbf{i} + \left(\frac{c\cos\eta}{\cosh\xi-\cos\eta} - \frac{c\sin^2\eta}{(\cosh\xi-\cos\eta)^2}\right)\mathbf{j}. \end{aligned}$$

So then we see that

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \xi}\cdot\frac{\partial \mathbf{r}}{\partial \eta} &= \frac{-c\sin\eta\sinh\xi}{(\cosh\xi-\cos\eta)^2}\left(\frac{c\cosh\xi}{\cosh\xi-\cos\eta} - \frac{c\sinh^2\xi}{(\cosh\xi-\cos\eta)^2} + \frac{c\cos\eta}{\cosh\xi-\cos\eta} - \frac{c\sin^2\eta}{(\cosh\xi-\cos\eta)^2}\right) \\ &= \cdots = 0, \end{aligned}$$

so that the system is indeed orthogonal.

To find the scale factors, first simplify

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{c\,\cosh^2\xi - c\,\cosh\xi\cos\eta - c\,\sinh^2\xi}{(\cosh\xi-\cos\eta)^2} \\ &= \frac{c(1-\cosh\xi\cos\eta)}{(\cosh\xi-\cos\eta)^2}. \end{aligned}$$

Then

$$\begin{aligned} \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 &= \frac{c^2}{(\cosh\xi-\cos\eta)^4}\left\{(1-\cosh\xi\cos\eta)^2 + \sin^2\eta\sinh^2\xi\right\} \\ &= \frac{c^2}{(\cosh\xi-\cos\eta)^4}(\cosh\xi-\cos\eta)^2 \\ &= \frac{c^2}{(\cosh\xi-\cos\eta)^2} \end{aligned}$$

Hence

$$h_1 = \left|\frac{\partial \mathbf{r}}{\partial \xi}\right| = \sqrt{\left(\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right)} = \frac{c}{(\cosh\xi-\cos\eta)}.$$

Similarly:

$$h_2 = \left|\frac{\partial \mathbf{r}}{\partial \eta}\right| = \sqrt{\left(\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right)} = \cdots = \frac{c}{(\cosh\xi - \cos\eta)}.$$

Finally:

$$h_3 = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1.$$

4. First we work out

$$\partial\mathbf{r}/\partial u = u\mathbf{i} + v\mathbf{j} = h_1\widehat{\mathbf{e}}_1, \;\; \partial\mathbf{r}/\partial v = -v\mathbf{i} + u\mathbf{j} = h_2\widehat{\mathbf{e}}_2.$$

Then

$$h_1 = h_2 = \left|\frac{\partial \mathbf{r}}{\partial u}\right| = \left|\frac{\partial \mathbf{r}}{\partial v}\right| = (u^2+v^2)^{1/2}, \;\; h_3 = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1.$$

It then follows that

$$\widehat{\mathbf{e}}_1 = (u\mathbf{i} + v\mathbf{j})/(u^2+v^2)^{1/2}, \;\; \widehat{\mathbf{e}}_2 = (-v\mathbf{i} + u\mathbf{j})/(u^2+v^2)^{1/2}, \;\; \widehat{\mathbf{e}}_3 = \mathbf{k}.$$

5. (i) Using our expression for div in curvilinear coordinates, with the values of h_1, h_2, h_3 calculated in the previous question, along with $F_1 = u(u^2+v^2)^{3/2}, F_2 = -v(u^2+v^2)^{3/2}, F_3 = 0$, we have

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u}(h_2h_3F_1) + \frac{\partial}{\partial v}(h_3h_1F_2)\right)\\
&= \frac{1}{(u^2+v^2)}\left(\frac{\partial}{\partial u}(u(u^2+v^2)^2 - \frac{\partial}{\partial v}(v(u^2+v^2)^2\right)\\
&= \frac{1}{(u^2+v^2)}\left((u^2+v^2)^2 + 4u^2(u^2+v^2) - (u^2+v^2)^2 - 4v^2(u^2+v^2)\right)\\
&= 4(u^2-v^2), \text{ as required.}
\end{aligned}$$

(ii) Using the curvilinear formula for curl:

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\widehat{\mathbf{e}}_1 & h_2\widehat{\mathbf{e}}_2 & h_3\widehat{\mathbf{e}}_3 \\ \partial/\partial u & \partial/\partial v & \partial/\partial z \\ h_1F_1 & h_2F_2 & 0 \end{vmatrix} = \frac{\widehat{\mathbf{e}}_3}{(u^2+v^2)}\left(\frac{\partial}{\partial u}(-v(u^2+v^2)^2 - \frac{\partial}{\partial v}(u(u^2+v^2)^2\right)\\
&= \widehat{\mathbf{e}}_3(-4uv - 4uv) = -8uv\widehat{\mathbf{e}}_3.
\end{aligned}$$

(iii) Using the expressions found for the unit vectors in Q4, we have

$$\begin{aligned}
\mathbf{F} &= u(u^2+v^2)^{3/2}\widehat{\mathbf{e}}_1 - v(u^2+v^2)^{3/2}\widehat{\mathbf{e}}_2\\
&= [u(u^2+v^2)^{3/2}(u\mathbf{i}+v\mathbf{j}) - v(u^2+v^2)^{3/2}(-v\mathbf{i}+u\mathbf{j})]/(u^2+v^2)^{1/2}\\
&= \mathbf{i}(u^2+v^2)(u^2+v^2)\\
&= \mathbf{i}[(u^2-v^2)^2 + 4u^2v^2]\\
&= \mathbf{i}(4x^2+4y^2),
\end{aligned}$$

as required. Given the Cartesian form of $\mathbf{F}$ we can then calculate that

$$\operatorname{div}\mathbf{F} = \partial/\partial x(4x^2+4y^2) = 8x = 4(u^2-v^2),$$

which agrees with the earlier calculation. We can also calculate

$$\operatorname{curl}\mathbf{F} = -\mathbf{k}\partial/\partial y(4x^2+4y^2) = -8y\mathbf{k} = -8uv\mathbf{k},$$

which agrees with the previous result since $\widehat{\mathbf{e}}_3 = \mathbf{k}$.

6. We have

$$\widehat{\mathbf{r}} = \cos\phi\,\mathbf{i} + \mathbf{j}\,\sin\phi, \widehat{\phi} = -\mathbf{i}\,\sin\phi + \mathbf{j}\,\cos\phi, \widehat{\mathbf{z}} = \mathbf{k}.$$

Rearranging:

$$\mathbf{i} = \widehat{\mathbf{r}}\cos\phi - \widehat{\phi}\sin\phi, \mathbf{j} = \widehat{\mathbf{r}}\sin\phi + \widehat{\phi}\cos\phi.$$

Since $x = r\cos\phi, y = r\sin\phi$ we have

$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} = r\sin\phi(\widehat{\mathbf{r}}\cos\phi - \widehat{\phi}\sin\phi) + z(\widehat{\mathbf{r}}\sin\phi + \widehat{\phi}\cos\phi) + (r\cos\phi)\mathbf{k},$$

so that

$$F_r = r\sin\phi\cos\phi + z\sin\phi, \quad F_\phi = -r\sin^2\phi + z\cos\phi, \quad F_z = r\cos\phi.$$

7. We need to work out the Jacobian determinant

$$\det J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -2x & 2y \\ 2y & 2x \end{vmatrix} = -4(x^2+y^2).$$

Then we know that $|\det J|\; dx\, dy = du\, dv \Rightarrow 4(x^2+y^2)\, dx\; dy = du\, dv$. Also note that $(x^2+y^2)^2 = u^2+v^2$. Then:

$$\int_R (x^2+y^2)^3 dx\, dy = \int_R (x^2+y^2)^2 \frac{1}{4}\, du\, dv = \frac{1}{4}\int_R (u^2+v^2)\, du\, dv.$$

We now have to work out the limits in terms of u and v. The region is bounded by $x^2 - y^2 = 1 \Rightarrow u = -1$ and $y^2 - x^2 = 1 \Rightarrow u = 1$, together with $xy = 1, xy = 2$ which translate to $v = 2$ and $v = 4$. The required integral is therefore

$$\begin{aligned} & \frac{1}{4}\int_{v=2}^{v=4}\int_{u=-1}^{u=1} (u^2+v^2)\, du\, dv \\ = & \frac{1}{4}\int_{v=2}^{v=4} \left[\frac{u^3}{3} + v^2 u\right]_{u=-1}^{u=1} dv \\ = & \frac{1}{4}\int_{v=2}^{v=4} \left(\frac{2}{3} + 2v^2\right) dv \\ = & \cdots = 29/3. \end{aligned}$$

8. In plane polars (r,θ) we have $x = r\cos\theta, y = r\sin\theta$. The relevant Jacobian determinant is

$$\det J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus we have $dx\, dy = r\, dr\, d\theta$ (a result we have made use of in earlier questions). We also have $x^4 + y^4 = r^4\cos^4\theta + r^4\sin^4\theta$. In plane polars the circular disc is the region $0 \le r \le 1, 0 \le \theta \le 2\pi$. The required integral transforms to

$$\begin{aligned} \int_0^{2\pi}\int_0^1 (r^4\cos^4\theta + r^4\sin^4\theta) r\, dr\, d\theta &= \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\left[\frac{r^6}{6}\right]_0^1 d\theta \\ &= \frac{1}{6}\int_0^{2\pi} (1 - 2\cos^2\theta\sin^2\theta)\, d\theta \\ &= \frac{1}{6}\int_0^{2\pi} \left(\frac{3}{4} + \frac{1}{4}\cos 4\theta\right) d\theta \\ &= \pi/4. \end{aligned}$$

9. The Jacobian determinant is

$$\det J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Thus we have $2dx\, dy = du\, dv$. We note that the integrand $(x+y)^2\cos(x^2-y^2)$ can be written as $v^2\cos uv$. It's also useful to observe that $u+v = 2x$ and $v-u = 2y$. The boundaries of the region therefore become

$u = \pm v$ (which intersect at $v = 0$) and $v = 1$. The integral is transformed to

$$\begin{aligned} \int_0^1 \int_{-v}^{v} v^2 \cos(uv)\, \frac{1}{2} du\, dv &= \frac{1}{2}\int_0^1 \left[v\sin(uv)\right]_{u=-v}^{u=v}\, dv \\ &= \int_0^1 v \sin v^2 dv \\ &= \frac{1}{2}\int_0^1 \sin t\, dt = \frac{1}{2}(1-\cos(1)). \end{aligned}$$

10. We calculate

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \lambda} &= (\frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}, \frac{\partial z}{\partial \lambda}) = (\cos s, \sin s, 0) \\ \frac{\partial \mathbf{r}}{\partial s} &= (\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}) = (-\lambda \sin s, \lambda \cos s, 1), \end{aligned}$$

and so

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \lambda} \times \frac{\partial \mathbf{r}}{\partial s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos s & \sin s & 0 \\ -\lambda \sin s & \lambda \cos s & 1 \end{vmatrix} = (\sin s, -\cos s, \lambda).$$

Thus $|\mathbf{J}| = \sqrt{1+\lambda^2}$ and hence

$$S = \int_S dS = \int_S |\mathbf{J}|\, d\lambda\, ds = \int_{s=0}^{2\pi}\int_{\lambda=0}^{1} (1+\lambda^2)^{1/2} d\lambda\, ds.$$

Solve using the substitution $\lambda = \sinh t$ to get

$$S = 2\pi \left[\frac{1}{2}t + \frac{1}{4}\sinh 2t\right]_0^{t_1}, \text{ where } \sinh t_1 = 1.$$

Finally, $\sinh 2t_1 = 2\sinh t_1 \cosh t_1 = 2 \sinh t_1 \sqrt{1+\sinh^2 t_1} = 2\sqrt{2}$, and so

$$S = \pi(\sinh^{-1}(1) + \sqrt{2}).$$

11. We start by calculating

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial t} &= (-b\sin t\cos\theta, -b\sin t\sin\theta, b\cos t) \\ \frac{\partial \mathbf{r}}{\partial \theta} &= (-(a+b\cos t)\sin\theta, (a+b\cos t)\cos\theta, 0), \end{aligned}$$

and then

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \theta} = (-(a+b\cos t)(b\cos t\cos\theta), (a+b\cos t)(b\cos t\sin\theta), -(a+b\cos t)(b\sin t)(\cos^2\theta + \sin^2\theta))$$

and so

$$|\mathbf{J}| = b(a+b\cos t)\sqrt{\cos^2 t\cos^2\theta + \cos^2 t\sin^2\theta + \sin^2 t} = b(a+b\cos t).$$

The required integral is

$$\begin{aligned} \int_S z^2\, dS &= \int_{\theta=0}^{2\pi}\int_{t=0}^{2\pi} (b^3\sin^2 t)(a+b\cos t)\, dt\, d\theta \\ &= 2\pi b^3 \int_0^{2\pi} (a\sin^2 t + b\sin^2 t\cos t)\, dt \\ &= 2\pi^2 a b^3. \end{aligned}$$