Problem Sheet 4

1. A change of variables is given by

$$x = u^3 + uv + v^3, \quad y = u^2 - v^2.$$

Find (x, y) when $(u, v) = (1, 0)$. Use the inverse function theorem to show that in the neighbourhood of this point, u and v are single-valued differentiable functions of x and y (although these functions cannot be written down explicitly). Calculate the partial derivatives of u and v with respect to x and y at this point.

2.(i) The variables (u, v) are expressed implicitly in terms of (x, y) as follows:

$$u^2 + 2v^2 + y^2 - x^2 = 6, \quad uvy - vxy = 1.$$

Suppose that these equations are satisfied by $(x, y, u, v) = (x_0, y_0, u_0, v_0)$. By differentiating the above equations implicitly with respect to x and y, show that $\partial u/\partial x, \partial v/\partial x, \partial u/\partial y, \partial v/\partial y$ all exist at (x_0, y_0, u_0, v_0) provided

$$\det \begin{pmatrix} 2u_0 & 4v_0 \\ v_0 y_0 & u_0 y_0 - x_0 y_0 \end{pmatrix} \neq 0.$$

Show that two possible solutions for (x_0, y_0, u_0, v_0) are $(1, 1, 2, 1)$ and $(1 + \sqrt{2}, -1 - \sqrt{2}, 2, 1)$. Show that the determinant condition is satisfied at one point but not at the other.

(ii) Consider the same equations but this time write them as a map (from $\mathbb{R}^4$ to $\mathbb{R}^2$) given by

$$F(x, y, u, v) = (u^2 + 2v^2 + y^2 - x^2 - 6, uvy - vxy - 1) = 0.$$

Show that the determinant condition derived in (i) is equivalent to the non-vanishing of the determinant of the Jacobian matrix

$$\begin{pmatrix} \partial F_1/\partial u & \partial F_1/\partial v \\ \partial F_2/\partial u & \partial F_2/\partial v \end{pmatrix},$$

at (x_0, y_0, u_0, v_0) where F_1, F_2 are the components of F.

The general form of this result is known as the ***Implicit Function Theorem*** *(a consequence of the inverse function theorem) and will be seen in more detail in Differential Equations in Term 2.*

(iii) Would the variables (u, v) as defined above in terms of x and y be a useful alternative coordinate system to the Cartesian one?

3. Bipolar coordinates (ξ, η, z) are given in terms of Cartesian coordinates (x, y, z) by

$$x = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \quad y = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad z = z,$$

where c is a constant. Show that this coordinate system is orthogonal, and find the scale factors h_1, h_2, h_3.

4. Consider the following curvilinear coordinate system (u, v, z) defined in terms of the Cartesian coordinates (x, y, z) as:

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z,$$

with $u \geq 0$. Find the scale factors h_1, h_2, h_3 and express the unit vectors $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3$ of the curvilinear coordinate system in terms of the Cartesian unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

5. The vector field $\mathbf{F}$ is given in terms of the curvilinear coordinate system described in Q4 as

$$\mathbf{F} = u(u^2 + v^2)^{3/2} \widehat{\mathbf{e}}_1 - v(u^2 + v^2)^{3/2} \widehat{\mathbf{e}}_2.$$

Show that

$$\begin{aligned} &\text{(i)} && \operatorname{div} \mathbf{F} = 4(u^2 - v^2), \\ &\text{(ii)} && \operatorname{curl} \mathbf{F} = -8uv\, \widehat{\mathbf{e}}_3, \\ &\text{(iii)} && \mathbf{F} = 4(x^2 + y^2)\, \mathbf{i}. \end{aligned}$$

Using the result in (iii), confirm your answers to (i) and (ii) by carrying out the calculations in Cartesian coordinates.

6. Find the unit vectors $\widehat{\mathbf{r}}, \widehat{\phi}, \widehat{\mathbf{z}}$ of a cylindrical polar coordinate system (r, ϕ, z) in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Solve for $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in terms of $\widehat{\mathbf{r}}, \widehat{\phi}, \widehat{\mathbf{z}}$. Represent the vector

$$\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\mathbf{k}$$

in cylindrical coordinates and determine F_r, F_ϕ and F_z.

7. Using the substitution

$$u = y^2 - x^2, \;\; v = 2xy,$$

evaluate the integral

$$\int_R (x^2 + y^2)^3 dx\, dy$$

where R is the finite region in the first quadrant, bounded by the curves $x^2 - y^2 = 1, y^2 - x^2 = 1, xy = 1$ and $xy = 2$.

8. Use plane polar coordinates to evaluate the integral

$$\int_R (x^4 + y^4)\, dx\, dy$$

where R is the circular disc $x^2 + y^2 \le 1$.

9. Use the transformation

$$u = x - y, v = x + y,$$

to evaluate

$$\int_R (x + y)^2 \cos(x^2 - y^2)\, dx\, dy$$

where R is the region in the $x - y$ plane enclosed by the lines $y = 0, x = 0$ and $y = 1 - x$.

10. Find the surface area of the parameterized helicoid

$$x = \lambda \cos s, \;\; y = \lambda \sin s, \;\; z = s,$$

with $0 \leqslant \lambda \leqslant 1$ and $0 \leqslant s \leqslant 2\pi$.

11. Find

$$\int_S z^2\, dS,$$

where S is the surface of a torus, parameterized as

$$x = (a + b\cos t)\cos\theta, \;\; y = (a + b\cos t)\sin\theta, \;\; z = b\sin t,$$

with $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant t \leqslant 2\pi$ and $a > b > 0$.

Sheet 4 Answers

1. $(x, y) = (1, 1)$, $u_x = 0, v_x = 1, u_y = 1/2, v_y = -3/2$.
2. (iii) No (explain why not!)
3. $h_1 = c/(\cosh\xi - \cos\eta) = h_2; h_3 = 1$.
4. $h_1 = (u^2 + v^2)^{1/2} = h_2; h_3 = 1; \widehat{\mathbf{e}}_1 = (u\mathbf{i} + v\mathbf{j})/(u^2 + v^2)^{1/2}, \widehat{\mathbf{e}}_2 = (-v\mathbf{i} + u\mathbf{j})/(u^2 + v^2)^{1/2}, \widehat{\mathbf{e}}_3 = \mathbf{k}$.
6. $\widehat{\mathbf{r}} = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi, \widehat{\phi} = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi, \widehat{\mathbf{z}} = \mathbf{k}$;
   $\mathbf{i} = \widehat{\mathbf{r}}\cos\phi - \widehat{\phi}\sin\phi, \mathbf{j} = \widehat{\mathbf{r}}\sin\phi + \widehat{\phi}\cos\phi$;
   $F_r = r\sin\phi\cos\phi + z\sin\phi, F_\phi = -r\sin^2\phi + z\cos\phi, F_z = r\cos\phi$.
7. $29/3$.
8. $\pi/4$.
9. $\frac{1}{2}(1 - \cos(1))$.
10. $\pi(\sinh^{-1}(1) + \sqrt{2})$.
11. $2\pi^2 ab^3$.