Sheet 5 Solutions

1. For $x > 0$ we have $q'(x) = (1/x^2)e^{-1/x}$, $q'' = -(2/x^3)e^{-1/x} + (1/x^4)e^{-1/x}$ etc. We can see that each derivative is continuous except perhaps at $x = 0$. However the function decays exponentially as $x \to 0$ so that $0 = \lim_{x\to 0+} q = \lim_{x\to 0+} q' = \lim_{x\to 0+} q''$ etc. Therefore since $q = 0$ for $x = 0$ the derivatives are indeed continuous at $x = 0$. For $x < 0$ we have $q(x) \equiv 0$ and so the function is trivially smooth in this region.
Since h is just a product of two smooth functions it is also smooth and it is easy to see that it is only 'switched on' when $0 < x < 1$. When it is 'switched on' it is the product of two exponential functions and so is always positive and hence $\int_0^1 h(x)\,dx > 0$.
To change the interval to (x_1, x_2) we simply consider the function $h((x - x_1)/(x_2 - x_1))$.

2. We have $y = x^3 + \varepsilon \sin 2\pi x \Longrightarrow y' = 3x^2 + 2\pi\varepsilon \cos 2\pi x$
$\Rightarrow I = \int_A^B 12x^4 + 12\varepsilon x \sin 2\pi x + 9x^4 + 4\pi\varepsilon^2 \cos^2 2\pi x + 12\varepsilon\pi x^2 \cos 2\pi x\,dx$
$\Rightarrow I'(0) = \int_0^1 12x \sin 2\pi x + 12\pi x^2 \cos 2\pi x\,dx$
$=$ (by parts) $= \int_0^1 12x \sin 2\pi x + [6x^2 \sin 2\pi x]_0^1 - \int_0^1 12x \sin 2\pi x\,dx = 0.$
Thus I is staionary and the extremal curve is $y = x^3$.
Consider $f = 12xy + (y')^2$. Then $\partial f/\partial y = 12x,\ \partial f/\partial y' = 2y'$.
Then the E-L equation is $12x - 2y'' = 0 \Rightarrow y' = 3x^2 + c \Rightarrow y = x^3 + cx + d$.
Applying $y(0) = 0, y(1) = 1 \Rightarrow c = d = 0$ and hence $y = x^3$.
The stationary value of I is $\int_0^1 12x^4 + 9x^4\,dx = 21/5$.

3. Let $f(x, y, y') = 2xyy' + y'^2 \Rightarrow \partial f/\partial y = 2xy',\ \partial f/\partial y' = 2xy + 2y'$.
Subst into E-L equation to get $2xy' - (2y + 2xy' + 2y'') = 0 \Rightarrow y'' + y = 0 \Rightarrow y = A \sin x + B \cos x$.
Boundary conditions: $y(0) = 0 \Rightarrow B = 0, y(\pi/2) = 1 \Rightarrow A = 1 \Rightarrow y = \sin x$.

4. Applying the end conditions we see that $x_1 = \beta \cosh(\gamma/\beta),\ x_2 = \beta \cosh((y_2 - \gamma)/\beta)$. If x_1 is small then we require β to be small, since $\cosh(\gamma/\beta) \geq 1$. But if β is small we also need γ small, otherwise the cosh term will become large. By a similar argument we also need $(y_2 - \gamma)$ to be small but this is not possible since γ is small and y_2 is assumed large. This means that if we have a surface linking two discs and continue to separate the discs, the surface will eventually break as can be seen in the video.

5. Let $f(r, \theta, \theta') = r^2(1 + r^2\theta'^2)^{1/2}$, independent of θ.
Then the E-L equation reduces to $\partial f/\partial\theta' =$ constant $\Rightarrow r^4\theta'/(1 + r^2\theta'^2)^{1/2} = c_1$.
Rearranging: $\theta'^2 = c_1^2/(r^2(r^6 - c_1^2)) \Rightarrow \theta = \int c_1/(r(r^6 - c_1^2)^{1/2})\,dr +$ constant.
Make substitution $r^3 = c_1 \sec u \Rightarrow 3r^2dr = c_1 \sec u \tan u\,du \Rightarrow (dr/r) = (1/3)\tan u\,du$
and $(r^6 - c_1^2)^{1/2} = c_1(\sec^2 u - 1)^{1/2} = c_1 \tan u$.
Therefore $\theta = (1/3)u +$ constant $\Rightarrow \sec(3\theta + c_2) = \sec u = r^3/c_1 \Rightarrow r^3 = c_1 \sec(3\theta + c_2)$.

6. Recall that in spherical polars: $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2$.
If we are constrained to the surface of a sphere of radius 1 then $r = 1$ and $dr = 0$
so that $ds = ((d\theta)^2 + \sin^2\theta(d\phi)^2)^{1/2}$ and so $L = \int ds = \int (1 + \sin^2\theta(d\phi/d\theta)^2)^{1/2}d\theta$, as required.
Let $f(\theta, \phi, \phi') =.(1 + \sin^2\theta(\phi')^2)^{1/2}$, which is independent of ϕ.
The E-L equation therefore reduces to $\partial f/\partial\phi' =$ constant
$\Rightarrow \phi' \sin^2\theta(1 + \sin^2\theta(\phi')^2)^{-1/2} = K$. Rearranging: $\phi' = K \csc\theta/(\sin^2\theta - K^2)^{1/2}$.
Multiply top and bottom by $\csc\theta$: $\phi' = K \csc^2\theta/(1 - K^2 \csc^2\theta)^{1/2} \equiv K \csc^2\theta/((1 - K^2) - K^2 \cot^2\theta)^{1/2}$
Integrating and setting $u = \cot\theta, du = -\csc^2\theta\,d\theta$ we have
$\phi = -\int K/((1 - K^2) - K^2u^2)^{1/2}du = -\sin^{-1}(Ku/(1 - K^2)^{1/2}) + \alpha$
$\Rightarrow \sin(\alpha - \phi) = Ku/(1 - K^2)^{1/2} = \beta u$, say. Thus $\beta \cot\theta = \sin(\alpha - \phi)$ as required.

7. Define $v(x) = f(x) - \lambda g(x)$. Then $v(x)$ is continuous.
Now $\int_{x_1}^{x_2} g(x)v(x)\,dx = \int_{x_1}^{x_2} g(x)f(x)\,dx - \lambda \int_{x_1}^{x_2}(g(x))^2dx = 0$, in view of the definition of λ.
Therefore $\int_{x_1}^{x_2} f(x)v(x)\,dx = 0$. However, $\int_{x_1}^{x_2} f(x)v(x)\,dx = \int_{x_1}^{x_2}(v(x))^2dx + \lambda \int_{x_1}^{x_2} g(x)v(x)\,dx = \int_{x_1}^{x_2}(v(x))^2dx$.
Thus $\int_{x_1}^{x_2}(v(x))^2dx = 0$ and hence $v(x) \equiv 0 \Rightarrow f(x) = \lambda g(x)$.

8. Apply the E-L equation to $f(y, y') = y'^2 + \lambda y$ (independent of x).
$\Rightarrow f - y'\partial f/\partial y' =$ constant, i.e. $y'^2 + \lambda y - 2y'^2 =$ constant.
Integrating: $\int \frac{dy}{(\lambda y+k)^{1/2}} = \int dx \Rightarrow x + c = (2/\lambda)(\lambda y + k)^{1/2}$.
Boundary conditions $y(0) = 0 \Rightarrow c^2 = 4k/\lambda^2$. $y(-1) = 0 \Rightarrow c = 1/2$.

Thus $y = (\lambda x/4)(x+1)$. To find λ substitute into $\int_0^1 y\,dx = 1$.
$\Rightarrow (\lambda/4)\int_0^1 x^2 + x\,dx = 1 \Rightarrow \lambda = 24/5 \Rightarrow y = (6/5)x(x+1)$.
Therefore $I = \int_0^1 y'^2 dx = (36/25)\int_0^1 (2x+1)^2 dx = 156/25$.

9. Let $f(x,y,y') = x^2y'^2 + 2y^2$. The E-L equation becomes $4y - \frac{d}{dx}(2x^2y') = 0$
$\Rightarrow x^2y'' + 2xy' - 2y = 0$. This is a Cauchy-Euler type ode which can be reduced to constant coefficients by the substitution $x = e^s \Rightarrow d^2y/ds^2 + dy/ds - 2y = 0 \Rightarrow y = Ae^{-2s} + Be^s \Rightarrow y(x) = Bx + A/x^2$.
Applying end conditions: $y(1) = 0 \Rightarrow A + B = 0$; $y(2) = 1 \Rightarrow 1 = 2B + A/4 \Rightarrow A = -4/7, B = 4/7$
$\Rightarrow y(x) = (4/7)(x - 1/x^2)$, as required.
To impose the constraint apply the E-L equation to $f = x^2y'^2 + 2y^2 + \lambda y/x$.
E-L equation becomes $x^2y'' + 2xy' - 2y = \lambda/2x$.
As above, the solution of the homogeneous equation is $Bx + A/x^2$.
To find the particular solution again use the substitution $x = e^s$ so that the ode becomes
$d^2y/ds^2 + dy/ds - 2y = (1/2)\lambda e^{-s}$.
For the particular solution try $y = \beta e^{-s}$ and substitute in to find $\beta = -\lambda/4$.
Solution is therefore $y(x) = Bx + A/x^2 - \lambda/4x$.
Apply end conditions to get $A = -4/7 + 3\lambda/14, B = 4/7 + \lambda/28$.
(Note that the values for A, B agree with those obtained previously when $\lambda = 0$).
To find λ substitute into integral constraint: $1/4 = \int_1^2 (B + A/x^3 - \lambda/4x^2)dx = B + 3A/8 - \lambda/8$.
Substitute in values for $A, B: 1/4 = 4/7 + \lambda/28 + (3/8)(-4/7 + 3\lambda/14) - \lambda/8 \Rightarrow \lambda = 12$
$\Rightarrow A = 2, B = 1$. Hence the new extremal curve is $y = 2/x^2 + x - 3/x$.

10. Let $f(x,y,y') = m^2y^2 - y'^2 + \lambda y\cos nx$.
The E-L equation yields $2m^2y + \lambda\cos nx + 2y'' = 0 \Rightarrow y'' + m^2y = -(\lambda/2)\cos nx$.
The complementary solution is $A\cos mx + B\sin mx$.
For the particular solution try $C\cos nx + D\sin nx$ assuming $m \neq n$.
Upon substitution we find that $D = 0, C = -(\lambda/2)/(m^2 - n^2)$.
Thus the general solution is $y = A\cos mx + B\sin mx - (\lambda/2)/(m^2 - n^2)\cos nx$.
Applying the end conditions: $y(0) = 1 \Rightarrow 1 = A - (\lambda/2)/(m^2 - n^2)$. $y'(2\pi) = \pi/2 \Rightarrow B = \pi/2m$.
So we have $y = A\cos mx + (\pi/2m)\sin mx + (1 - A)\cos nx$. To find A use integral constraint.
$\pi/2 = \int_0^{2\pi} A\cos mx\cos nx + (\pi/2m)\sin mx\cos nx + (1-A)\cos^2 nx\,dx = (1-A)\pi \Rightarrow A = 1/2$.
The solution when $m \neq n$ is therefore $y = (1/2)(\cos mx + \cos nx) + (\pi/2m)\sin mx$.
To determine what happens when $m = n$ we need to go back to the ode $y'' + m^2y = -(\lambda/2)\cos mx$.
The homogeneous solution is as above but now the RHS is contained in that solution.
This means we need to modify our trial function for the particular solution.
We try $y_p = Cx\cos mx + Dx\sin mx$ and find that $C = 0, D = -\lambda/4m$.
Thus the general solution now is $y = A\cos mx + B\sin mx - (\lambda/4m)x\sin mx$.
The boundary conditions fix the constants as $A = 1, B = (\pi/2m)(1 + \lambda)$.
We now substitute into the integral constraint to find λ :
$\pi/2 = \int_0^{2\pi} \cos^2 mx + (\pi/2m)(1+\lambda)\sin mx\cos mx - (\lambda/4m)x\sin mx\cos mx\,dx$
$\Rightarrow \pi/2 = \pi + 0 - (\lambda/8m)\int_0^{2\pi} x\sin 2mx\,dx$.
Integrating by parts: $\pi/2 = (\lambda/8m)\left[-(x/2m)\cos 2mx\right]_0^{2\pi} + (\lambda/8m)(1/2m)\int_0^{2\pi}\cos 2mx\,dx$.
The final integral is zero,leaving $\pi/2 = -\pi\lambda/8m^2 \Rightarrow \lambda = -4m^2$.
Finally, the solution for y when $m = n$ is $y = \cos mx + (\pi/2m)(1 - 4m^2)\sin mx + mx\sin mx$.

11. Writing the minimal surface equation in Cartesian coordinates (x,y) we have
$\partial/\partial x(f_x/(1 + f_x^2 + f_y^2)^{1/2}) + \partial/\partial y(f_y/(1 + f_x^2 + f_y^2)^{1/2}) = 0$
Expanding out and letting $g = 1 + f_x^2 + f_y^2$:
$(f_{xx} + f_{yy})/g^{1/2} + f_x(\partial/\partial x)g^{-1/2} + f_y(\partial/\partial y)g^{-1/2} = 0$
$\Rightarrow (f_{xx} + f_{yy})g^{-1/2} - f_x(f_xf_{xx} + f_yf_{xy})g^{-3/2} - f_y(f_xf_{xy} + f_yf_{yy})g^{-3/2} = 0$
$\Rightarrow g(f_{xx} + f_{yy}) - f_x^2f_{xx} - f_y^2f_{yy} - 2f_xf_yf_{xy} = 0$
$\Rightarrow (1 + f_x^2)f_{yy} + (1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} = 0$, as required.
If $f = ax + by + c$ then $f_{yy} = f_{xx} = f_{xy} = 0$ and so the equation is satisfied trivially.
If $f = \log(\cos x/\cos y) = \log(\cos x) - \log(\cos y)$ then $f_x = -\tan x$, $f_y = \tan y$ and $f_{xy} = 0$.
Also $f_{xx} = -\sec^2 x$, $f_{yy} = \sec^2 y$
and so $(1 + f_x^2)f_{yy} + (1 + f_y^2)f_{xx} = (1 + \tan^2 x)\sec^2 y - (1 + \tan^2 y)\sec^2 x = 0$.
Therefore this function also satisfies the minimal surface equation.