Problem Sheet 5

1. Show that the function

$$q(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

is smooth (i.e. infinitely differentiable) at every value of x. Hence deduce that the function

$$h(x) = q(x)q(1-x)$$

possesses the following properties:
(i) $h(x)$ is smooth for all x;
(ii) $h(x)$ is zero outside the interval $(0,1)$;
(iii) $\int_0^1 h(x)\,dx > 0$.

How can we modify h so that it vanishes outside an interval (x_1, x_2) with $\int_{x_1}^{x_2} h(x)\,dx > 0$?
(This is a suitable function for use in the proof of the 'Vanishing Lemma').

2. The equation $y = x^3 + \varepsilon \sin 2\pi x$ where ε is a parameter, describes a set of paths from the point $A(0,0)$ to the point $B(1,1)$. Substitute for y in the integral

$$I = \int_A^B (12xy + (y')^2)\,dx$$

and demonstrate explicitly that $dI/d\varepsilon = 0$ when $\varepsilon = 0$. Hence deduce that $y = x^3$ is an extremal curve of I which passes through A and B. Confirm this fact by solving the appropriate Euler-Lagrange equation and also calculate the corresponding stationary value of I.

3. Solve the Euler-Lagrange equation for the $y(x)$ which makes the integral

$$I = \int_0^{\pi/2} (2xyy' + (y')^2)\,dx$$

stationary, given $y = 0$ when $x = 0$ and $y = 1$ when $x = \pi/2$.

4. In lectures we showed that the minimal surface of revolution is given by rotating the curve

$$y = \pm\beta \cosh^{-1}(x/\beta) + \gamma$$

about the $y-$axis. If we suppose that $y(x_1) = 0$ and $y(x_2) = y_2$, write down expressions for x_1 and x_2 in terms of the other parameters. Show that no solution is possible if x_1 and x_2 are small but y_2 is large. To see what is happening physically see `https://www.youtube.com/watch?v=mziis4pbBOw`.

5. Find the extremal curves for the integral

$$I = \int r^2 \left(1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right)^{1/2} dr.$$

6. Show that the distance L between two points on a sphere of radius 1 can be written in the form

$$L = \int \left(1 + \left(\frac{d\phi}{d\theta}\right)^2 \sin^2\theta\right)^{1/2} d\theta.$$

Show that the extremal curve for this integral can be written in the form $\sin(\alpha - \phi) = \beta \cot\theta$. This curve is known as a 'great circle'.

7. Suppose f and g are continuous functions on $[x_1, x_2]$, and

$$\int_{x_1}^{x_2} \eta(x) f(x)\, dx = 0$$

for all continuous functions η such that

$$\int_{x_1}^{x_2} \eta(x) g(x)\, dx = 0.$$

Define

$$\lambda = \frac{\int_{x_1}^{x_2} f(x)g(x)\, dx}{\int_{x_1}^{x_2} (g(x))^2 dx}.$$

Show that

$$f(x) = \lambda g(x).$$

(This result is needed for deriving the Euler-Lagrange equation for problems involving constraints).

8. Find the stationary value of the integral

$$I = \int_0^1 (y')^2\, dx$$

subject to the constraint

$$J = \int_0^1 y\, dx = 1$$

and the end conditions $y(0) = 0$ and $y(-1) = 0$.

9. Show that the extremal curve $y = y(x)$ of the integral

$$I = \int_1^2 x^2 \left(\frac{dy}{dx}\right)^2 + 2y^2\, dx$$

which passes through the points $(1, 0)$ and $(2, 1)$ in the $x - y$ plane is given by

$$y = \frac{4}{7}\left(x - \frac{1}{x^2}\right).$$

If the constraint

$$\int_1^2 \frac{y}{x}\, dx = \frac{1}{4}$$

is added to the problem, find the new extremal curve of I.

10. Show that the extremal curve $y = y(x)$ of the integral

$$I = \int_0^{2\pi} m^2 y^2 - (y')^2 dx$$

satisfying the conditions

$$y(0) = 1, \;\; y'(2\pi) = \pi/2,$$
$$\int_0^{2\pi} y(x) \cos nx\, dx = \pi/2,$$

with m and n integers, is given by

$$y = \frac{1}{2}\left(\cos mx + \frac{\pi}{m} \sin mx + \cos nx\right)$$

provided $m \neq n$. Find the corresponding curve when $m = n$.

11. In lectures we showed that the minimal surface equation has the form

$$\operatorname{div}\left(\frac{\boldsymbol{\nabla} f}{(1+|\boldsymbol{\nabla} f|^2)^{1/2}}\right)=0.$$

where $f=f(x,y)$. Demonstrate that when expanded out this equation takes the form

$$(1+f_y^2)f_{xx}+(1+f_x^2)f_{yy}-2f_xf_yf_{xy}=0.$$

Show that both the plane $f=ax+by+c$ and the Scherk surface $f=\log(\cos x/\cos y)$ are solutions of this equation.

Answers

2. $I=21/5$;
3. $y=\sin x$;
5. $r^3=c_1\sec(3\theta-c_2)$;
8. $I=156/25$;
9. $y=2/x^2+x-3/x$;
10. $y=\cos mx+(\pi/2m)(1-4m^2)\sin mx+mx\sin mx$.