MATH50004 Differential Equations
Spring Term 2021/22
Problem Sheet 1

Exercise 1 (Variation of constants formula).

Let $a, g : \mathbb{R} \to \mathbb{R}$ be continuous functions, and consider an initial pair $(t_0, x_0) \in \mathbb{R}^2$. Find a solution $\lambda : \mathbb{R} \to \mathbb{R}$ of the initial value problem

$$\dot{x} = a(t)x + g(t)\,, \qquad x(t_0) = x_0\,,$$

and show that the solution you found is the unique solution on $\mathbb{R}$.

Hint. To find the solution, make the ansatz $\lambda(t) := c(t)\exp\big(\int_{t_0}^t a(s)\,\mathrm{d}s\big)$, and determine a differential equation for the function c. For uniqueness, use a similar argument as in Example 1.1.

Exercise 2 (No solution to an initial value problem).

Consider the one-dimensional initial value problem

$$\dot{x} = \begin{cases} 1 & : \quad x < 0 \\ -1 & : \quad x \geq 0 \end{cases}\,, \qquad x(0) = 0\,.$$

Show that this initial value problem has no solution.

Exercise 3 (Picard iterates).

Compute the first three Picard iterates $\lambda_1, \lambda_2, \lambda_3 : J \to \mathbb{R}^d$ corresponding to the two initial value problems

(i) $\dot{x} = x^2\,, \quad x(0) = 1\,,$

(ii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\,, \quad \begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

Note that for uniform convergence of the Picard iterates, J has to be chosen small enough, but you can do the computations here for $J = \mathbb{R}$.

Exercise 4 (Comparison to a solution of a differential inequality).

Let $I \subset \mathbb{R}$ be an interval and $\lambda : I \to \mathbb{R}$ be a solution of the differential equation

$$\dot{x} = f(t, x)\,,$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that a differentiable function $\alpha : I \to \mathbb{R}$ satisfies $\dot{\alpha}(t) > f(t, \alpha(t))$ for all $t \in I$, and we have $\alpha(t_0) \geq \lambda(t_0)$ for some $t_0 \in I$. Show that $\alpha(t) > \lambda(t)$ for all $t \in I$ with $t > t_0$.

Exercise 5 (Optional challenging question).

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Consider the differential equation

$$\dot{x} = f(x)\,.$$

Find out if there exists a solution $\lambda : \mathbb{R} \to \mathbb{R}$ and a $a > 0$ such that $\lambda(t + a) - \lambda(t) \in \mathbb{Z}$ for all $t \in \mathbb{R}$.

Hint. Use without proof that every initial value problem has a unique solution on $\mathbb{R}$ (we establish this later in the course). Distinguish if the function f has a zero or not.

Comments on importance and difficulty of the exercises. Exercise 1 is standard and concerns a very fundamental differential equation: a linear inhomogeneous differential equation. Linear equations will be discussed extensively in the course later, and the solution formula under consideration will be generalised to higher dimensions there under the assumption that the homogeneous part $a(t)x$ does not depend on time t. Exercise 2 can be solved readily, but a good understanding of how to argue in analysis is required. Exercise 3 concerns basic and standard material. Proving Exercise 4 rigorously is not so straightforward, although the arguments are very elementary. Try to first understand why the result is true. Exercise 5 is difficult, since it requires the combination of several insights that are probably not so natural to you with your current knowledge. Nevertheless, it is extremely useful to try to understand why such a statement holds, since it trains your thinking about differential equations. The statement itself has no consequences for what we do later, but understanding the proof is very helpful due to all the arguments being used in different scenarios later.