MATH50004 Differential Equations
Spring Term 2021/22
Solutions to Problem Sheet 2

Exercise 6.

(i) We verify the three conditions given in Definition 2.4 in both cases.

Firstly, clearly, $A = 0$ implies $\|A\| = 0$. On the other hand, if $A \neq 0$, then there exist $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ such that $a_{ij} \neq 0$. Consider the j-th unit vector e_j. Then $\frac{\|Ae_j\|}{\|e_j\|} = \|Ae_j\| > 0$, which proves positive definiteness. Concerning absolute homogeneity, let $b \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times m}$. Then

$$\|bA\| = \sup_{x\in\mathbb{R}^m\setminus\{0\}} \frac{\|bAx\|}{\|x\|} = \sup_{x\in\mathbb{R}^m\setminus\{0\}} |b|\frac{\|Ax\|}{\|x\|} = |b|\|A\|\,.$$

Given $A, B \in \mathbb{R}^{n\times m}$, the triangle inequality follows from

$$\begin{aligned}\|A+B\| &= \sup_{x\in\mathbb{R}^m\setminus\{0\}} \frac{\|(A+B)x\|}{\|x\|} = \sup_{x\in\mathbb{R}^m\setminus\{0\}} \frac{\|Ax\|+\|Bx\|}{\|x\|} \\ &\leq \sup_{x\in\mathbb{R}^m\setminus\{0\}} \frac{\|Ax\|}{\|x\|} + \sup_{x\in\mathbb{R}^m\setminus\{0\}} \frac{\|Bx\|}{\|x\|} = \|A\| + \|B\|\,.\end{aligned}$$

Concerning the max norm, positive definiteness is clear. Let $b \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times m}$. Then

$$\|bA\| = \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} |ba_{ij}| = \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} |b||a_{ij}| = |b| \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} |a_{ij}| = |b|\|A\|\,,$$

which proves positive homogeneity. Given $A, B \in \mathbb{R}^{n\times m}$, the triangle inequality follows from

$$\|A+B\| = \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} |a_{ij}+b_{ij}| \leq \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} (|a_{ij}|+|b_{ij}|) \leq \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} |a_{ij}| + \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} |b_{ij}| = \|A\| + \|B\|\,.$$

(ii) We prove this statement for $K_1 = 1$ and $K_2 = m\sqrt{n}$.

We first show $\|A\|_{\max} \leq \|A\|$ for all $A \in \mathbb{R}^{n\times m}$. Fix $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Then

$$|a_{ij}| \leq \sqrt{\sum_{\ell=1}^{n} a_{\ell j}^2} = \|Ae_j\| \leq \|A\|\,,$$

and taking the maximum over i and j implies the claim.

We now show $\|A\| \leq m\sqrt{n}\|A\|_{\max}$ for all $A \in \mathbb{R}^{n\times m}$. We have

$$\begin{aligned}\|A\| &= \max_{\|x\|=1} \|Ax\| \leq \max_{\|x\|=1} \left\| \|A\|_{\max} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} x \right\| \\ &\leq \|A\|_{\max} \max_{\|x\|=1} \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_1 \end{pmatrix} + \cdots + \begin{pmatrix} x_m \\ \vdots \\ x_m \end{pmatrix} \right\| \\ &\leq \|A\|_{\max} \max_{\|x\|=1} \left(\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_1 \end{pmatrix} \right\| + \cdots + \left\| \begin{pmatrix} x_m \\ \vdots \\ x_m \end{pmatrix} \right\| \right) \\ &\leq \|A\|_{\max} m \left\| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\| = m\sqrt{n}\|A\|_{\max}\,,\end{aligned}$$

which finishes the proof.

Exercise 7.

The function in (i) is the only non-differentiable function and requires a slightly different treatment. For differentiable functions, in the one-dimensional case, we can argue with the mean value theorem: a bounded derivative implies Lipschitz continuity, and an unbounded derivative implies that the function is not Lipschitz continuous. This argumentation goes as indicated in Example 2.6. In the higher-dimensional case, we only get the mean value inequality (see Theorem 2.8), which can be used only to prove Lipschitz continuity, but not to disprove it. Since all the higher-dimensional examples, given by (iv) and (v) turn out to be Lipschitz continuous, all results for (i)–(v) follow from understanding whether the derivative is bounded or unbounded.

(i) This function is not differentiable in 0, but in $[0,\infty)$ and $(-\infty,0]$. Since its derivative is equal to 1, it is Lipschitz continuous on each of these two domains. Consider now $x<0<y$. Then

$$|f(y)-f(x)| = |y+x| \le |y|+|x| = y-x = |y-x|\,,$$

so f is Lipschitz continuous with Lipschitz constant $K=1$ on $\mathbb{R}$.

(ii) f is not Lipschitz continuous, because it is differentiable on $(0,1]$, but the derivative $f'(x)=3x^{-\frac{2}{3}}$ is unbounded on $(0,1)$.

(iii) f is differentiable with $f'(x)=-\frac{1}{x^2}$, and the absolute value of its derivative is bounded by the Lipschitz constant $K=1$, so f is Lipschitz continuous.

(iv) Note that f is a linear function, and its constant derivative is given by $f'(x,y)=\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$, independent of $(x,y)\in\mathbb{R}^2$. Hence f' is bounded by the Lipschitz constant $\left\|\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}\right\|$, which can be estimated above by $K:=2\sqrt{2}2=4\sqrt{2}$ due to Exercise 6 (ii).

(v) We calculate $f'(x,y)=\left(-\frac{y(x^2-y^2-1)}{(x^2+y^2+1)^2},\frac{x(x^2-y^2+1)}{(x^2+y^2+1)^2}\right)$, and it is clear that $\|f'(x,y)\|$ is bounded on the compact set $\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 4\}$. Due to Exercise 6 (ii), a (non-optimal) upper bound is given by $\big\|(4\cdot(4^2-1),4\cdot(4^2+1))\big\|\le 2\cdot 4\cdot 17=102$. Note that the norm here is the (Euclidean) operator norm on $\mathbb{R}^{1\times 2}$, and not the Euclidean norm on $\mathbb{R}^2$.

Exercise 8.

Assume there exists a $\tilde{T}>t_0+h$ such that all solutions to the initial value problem $\dot{x}=f(t,x)$, $x(t_0)=x_0$, do not exist at time $\tilde{T}$. Define

$$T := \inf\big\{\tilde{T}>t_0+h : \text{all solutions satisfying the initial condition } x(t_0)=x_0 \text{ do not exist at time } \tilde{T}\big\}\,.$$

This implies that there exists a solution $\lambda:\big[t_0-h,T-\frac{h}{2}\big]\to\mathbb{R}^d$ of the initial value problem $\dot{x}=f(t,x)$, $x(t_0)=x_0$. Consider the initial value problem

$$\dot{x}=f(t,x)\,,\quad x\big(T-\tfrac{h}{2}\big)=\lambda\big(T-\tfrac{h}{2}\big)\,.$$

Theorem 2.11 implies that there exists a solution $\mu:\big[T-\frac{3h}{2},T+\frac{h}{2}\big]\to\mathbb{R}^d$ of this initial value problem. Since $\lambda(T-\frac{h}{2})=\mu(T-\frac{h}{2})$, Lemma 2.15 implies that both solutions coincide on the intersection of their domains, and λ can be extended up to time $t=T+\frac{h}{2}$ using the solution μ. This contradicts the definition of T, and thus there exist solutions of the initial value problems on intervals unbounded above. Similarly, one shows existence of solutions on intervals unbounded below.

Exercise 9.

(i) We verify the three conditions given in Definition 2.4 and consider first the supremum norm $\|\cdot\|_\infty$. Firstly, we get

$$\|u\|_\infty = 0 \Longleftrightarrow \sup_{t\in[a,b]} \|u(t)\| = 0 \Longleftrightarrow u(t) = 0 \text{ for all } t \in [a,b] \Longleftrightarrow u = 0\,.$$

Secondly, for $\alpha \in \mathbb{R}$, we have

$$\|\alpha u\|_\infty = \sup_{t\in[a,b]} \|\alpha u(t)\| = \sup_{t\in[a,b]} |\alpha|\|u(t)\| = |\alpha| \sup_{t\in[a,b]} \|u(t)\| = |\alpha|\|u\|_\infty\,.$$

Finally, for any $u, v \in C^0([a,b],\mathbb{R}^d)$, we get using the triangle inequality for the Euclidean norm and an elementary property of the supremum that

$$\begin{aligned}\|u+v\|_\infty &= \sup_{t\in[a,b]} \|u(t)+v(t)\| \le \sup_{t\in[a,b]} (\|u(t)\|+\|v(t)\|) \le \sup_{t\in[a,b]} \|u(t)\| + \sup_{t\in[a,b]} \|v(t)\| \\ &= \|u\|_\infty + \|v\|_\infty\,.\end{aligned}$$

Now consider the L^1-norm $\|\cdot\|_1$. Firstly,

$$\|u\|_1 = 0 \Longleftrightarrow \int_a^b \|u(t)\|\,\mathrm{d}t = 0 \overset{(*)}{\Longleftrightarrow} u(t) = 0 \text{ for all } t \in [a,b] \Longleftrightarrow u = 0\,.$$

$(*)$ requires some thought: while $(\Leftarrow)$ is clear, assume that $u(t_0) \neq 0$ for some $t_0 \in [a,b]$. Continuity of u then implies that there exists an $\varepsilon > 0$ and a $\delta > 0$ such that $\|u(t)\| \ge \varepsilon$ for all $t \in [t_0-\delta, t_0+\delta]$. Hence $\int_a^b \|u(t)\|\,\mathrm{d}t \ge \int_{t_0-\delta}^{t_0+\delta} \|u(t)\|\,\mathrm{d}t \ge \int_{t_0-\delta}^{t_0+\delta} \varepsilon\,\mathrm{d}t = 2\varepsilon\delta > 0$. That proves the implication $(\Rightarrow)$. Secondly,

$$\|\alpha u\|_1 = \int_a^b \|\alpha u(t)\|\,\mathrm{d}t = \int_a^b |\alpha|\|u(t)\|\,\mathrm{d}t = |\alpha| \int_a^b \|u(t)\|\,\mathrm{d}t = |\alpha|\|u\|_1\,,$$

and finally, for any $u, v \in C^0([a,b],\mathbb{R}^d)$, we get

$$\begin{aligned}\|u+v\|_1 &= \int_a^b \|u(t)+v(t)\|\,\mathrm{d}t \le \int_a^b (\|u(t)\|+\|v(t)\|)\,\mathrm{d}t = \int_a^b \|u(t)\|\,\mathrm{d}t + \int_a^b \|v(t)\|\,\mathrm{d}t \\ &= \|u\|_1 + \|v\|_1\,.\end{aligned}$$

(ii) The proof is divided into two steps.

Step 1. We show that the sequence $\{u_n\}_{n\in\mathbb{N}}$ from the hint is a Cauchy sequence with respect to $\|\cdot\|_1$. Consider two natural numbers n, m with $n \ge m$. Then we get

$$\big\|u_n - u_m\big\|_1 = \int_{-\frac{1}{m}}^{\frac{1}{m}} |u_n(t) - u_m(t)|\,\mathrm{d}t \overset{u_n(t),u_m(t)\in[-1,1]}{\le} \int_{-\frac{1}{m}}^{\frac{1}{m}} 2\,\mathrm{d}t = \frac{4}{m}\,. \tag{A}$$

Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\frac{4}{N} < \varepsilon$. Then (A) implies that we have that

$$\big\|u_n - u_m\big\|_1 \le \frac{4}{\min\{n,m\}} \le \frac{4}{N} < \varepsilon \quad \text{for all } n, m \ge N\,,$$

so the sequence $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Step 2. The sequence $\{u_n\}_{n\in\mathbb{N}}$ does not converge in the L^1 norm.

Assume to the contrary that there exists a continuous function $u_\infty \in C^0\big([a,b],\mathbb{R}^d\big)$ such that

$$\lim_{n\to\infty} \|u_n - u_\infty\|_1 = 0\,. \tag{B}$$

Obviously, $u_\infty(0) \neq 1$ or $u_\infty(0) \neq -1$. We only look at the case $u_\infty(0) \neq 1$, since the case $u_\infty(0) \neq -1$ can be treated analogously. Define $\varepsilon := \frac{1-u_\infty(0)}{2} > 0$. Due to continuity of the function u_∞, there exists a $\delta > 0$ such that

$$|u_\infty(t) - u_\infty(0)| < \varepsilon \quad \text{for all } t \in (-\delta, \delta)\,,$$

which implies in particular that

$$u_\infty(t) \leq 1 - \varepsilon \quad \text{for all } t \in (-\delta, \delta)\,.$$

Choose $N \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\delta}{2}$ for all $n \geq N$. Then

$$\|u_n - u_\infty\|_1 = \int_{-1}^{1} |u_n(t) - u_\infty(t)|\,\mathrm{d}t \geq \int_{\frac{\delta}{2}}^{\delta} |u_n(t) - u_\infty(t)|\,\mathrm{d}t \geq \int_{\frac{\delta}{2}}^{\delta} \varepsilon\,\mathrm{d}t = \frac{\varepsilon\delta}{2} \quad \text{for all } n \geq N\,.$$

This is a contradiction to (B) and finishes the proof.

Exercise 10.

The proof is divided in two steps.
Step 1. We show that the sequence of functions $\{\mu_n : J \to \mathbb{R}^d\}_{n\in\mathbb{N}_0}$*, where* $\mu_n(t) := f(t, \lambda_n(t))$ *for all* $t \in J$*, is uniformly convergent with limit* $\mu_\infty : J \to \mathbb{R}^d$*, where* $\mu_\infty(t) := f(t, \lambda_\infty(t))$ *for all* $t \in J$*.*
Consider the compact set

$$S := \big\{(t, x) \in J \times \mathbb{R}^d : \|x - \lambda_\infty(t)\| \leq 1\big\}\,.$$

Let $\varepsilon > 0$ arbitrarily. The continuous function f is uniformly continuous on the compact set S, which implies in particular that there exists a $\delta > 0$ such that

$$\|f(t, x) - f(t, y)\| < \varepsilon \quad \text{for all } (t, x), (t, y) \in S \text{ with } \|x - y\| < \delta\,. \tag{C}$$

Due to the uniform convergence of the sequence of functions $\big\{\lambda_n\big\}_{n\in\mathbb{N}}$ to λ_∞, there exists an $N \in \mathbb{N}$ such that

$$\|\lambda_n(t) - \lambda_\infty(t)\| < \delta \quad \text{for all } n \geq N \text{ and } t \in J\,,$$

and using (C), this implies

$$\|\mu_n(t) - \mu_\infty(t)\| = \|f(t, \lambda_n(t)) - f(t, \lambda_\infty(t))\| < \varepsilon \quad \text{for all } n \geq N \text{ and } t \in J\,,$$

which finishes Step 1.
Step 2. We show that λ_∞ *satisfies the integral equation* $\lambda_\infty(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_\infty(s))\,\mathrm{d}s$
First we show that

$$\lim_{n\to\infty} \int_{t_0}^{t} \mu_n(s)\,\mathrm{d}s = \int_{t_0}^{t} \lim_{n\to\infty} \mu_n(s)\,\mathrm{d}s = \int_{t_0}^{t} \mu_\infty(s)\,\mathrm{d}s\,, \tag{D}$$

which essentially needs uniform convergence of $\{\mu_n\}_{n\in\mathbb{N}}$. This can be seen as follows. We have

$$\int_{t_0}^{t} \|\mu_n(s) - \mu_\infty(s)\|_\infty\,\mathrm{d}s \geq \int_{t_0}^{t} (\mu_n(s) - \mu_\infty(s))\,\mathrm{d}s \geq -\int_{t_0}^{t} \|\mu_n(s) - \mu_\infty(s)\|_\infty\,\mathrm{d}s\,,$$

and since both left and right hand side of this inequality converge to 0 as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_{t_0}^{t} (\mu_n(s) - \mu_\infty(s))\,\mathrm{d}s = 0\,,$$

which proves (D). Now $\lambda_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\,\mathrm{d}s$ implies

$$\begin{aligned}
\lambda_\infty(t) &= \lim_{n\to\infty} \lambda_{n+1}(t) = \lim_{n\to\infty} \left(x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\,\mathrm{d}s\right) \\
&= x_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, \lambda_n(s))\,\mathrm{d}s \overset{\text{(D)}}{=} x_0 + \int_{t_0}^{t} f(s, \lambda_\infty(s))\,\mathrm{d}s\,,
\end{aligned}$$

which finishes the proof.