MATH50004 Differential Equations
Spring Term 2021/22
Problem Sheet 2

Exercise 6 (Matrix norms).

(i) Show that both the operator norm

$$\|A\| := \sup_{x\in\mathbb{R}^m\setminus\{0\}} \frac{\|Ax\|}{\|x\|} \quad \text{for all } A \in \mathbb{R}^{n\times m}$$

and the max norm

$$\|A\|_{\max} := \max_{\substack{i=1,\dots,n,\\ j=1,\dots,m}} |a_{ij}| \quad \text{for all } A = (a_{ij}) \in \mathbb{R}^{n\times m}$$

define norms on the space of real-valued $n \times m$ matrices $\mathbb{R}^{n\times m}$.

(ii) Suppose that the vector norms $\|\cdot\|$ in both $\mathbb{R}^n$ and $\mathbb{R}^m$ needed to define the operator norm in (i) are given by the Euclidean norms in these spaces. Show that there exists constants $0 < K_1 < K_2$ such that

$$K_1\|A\|_{\max} \le \|A\| \le K_2\|A\|_{\max} \quad \text{for all } A \in \mathbb{R}^{n\times m}\,.$$

Exercise 7 (Lipschitz continuity).

For each of the following functions, find a Lipschitz constant on the indicated domain, or prove that the function is not Lipschitz continuous:

(i) $f(x) = |x|$, where $x \in \mathbb{R}$,

(ii) $f(x) = x^{\frac{1}{3}}$, where $x \in [0, 1]$,

(iii) $f(x) = \frac{1}{x}$, where $x \in [1, \infty)$,

(iv) $f(x, y) = \begin{pmatrix} x + 2y \\ -y \end{pmatrix}$, where $(x, y) \in \mathbb{R}^2$,

(v) $f(x, y) = \frac{xy}{1+x^2+y^2}$, where $x^2 + y^2 \le 4$.

Exercise 8 (Global existence of solutions).

Show that the solution $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$ obtained in the global version of the Picard–Lindelöf theorem (Theorem 2.11) exists even globally, i.e. it can be extended to a solution defined on $\mathbb{R}$.

Exercise 9 (The space of continuous functions on a compact interval).

Consider the space of continuous functions $u : [a, b] \to \mathbb{R}^d$ defined on a compact interval $[a, b]$,

$$C^0\big([a, b], \mathbb{R}^d\big) := \big\{u : [a, b] \to \mathbb{R}^d : u \text{ is continuous}\big\}\,,$$

and consider both the supremum norm

$$\|u\|_\infty := \sup_{t\in[a,b]} \|u(t)\| \quad \text{for all } u \in C^0\big([a, b], \mathbb{R}^d\big)\,,$$

and the L^1 norm

$$\|u\|_1 := \int_a^b \|u(t)\|\,\mathrm{d}t \quad \text{for all } u \in C^0\big([a, b], \mathbb{R}^d\big)\,.$$

(i) Show that both norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ indeed define norms on the real vector space $C^0\big([a,b],\mathbb{R}^d\big)$.

(ii) The space $\big(C^0\big([a,b],\mathbb{R}^d\big),\|\cdot\|_\infty\big)$ is a Banach space. Show that this is not true for the space $\big(C^0\big([a,b],\mathbb{R}^d\big),\|\cdot\|_1\big)$.

<u>Hint.</u> Consider the sequence of functions $\{u_n\}_{n\in\mathbb{N}}$, where $u_n:[-1,1]\to\mathbb{R}$ is defined by

$$u_n(x) := \begin{cases} -1 & : \quad x \in \big[-1,-\frac{1}{n}\big]\,, \\ nx & : \quad x\in\big(-\frac{1}{n},\frac{1}{n}\big)\,, \\ 1 & : \quad x\in\big[\frac{1}{n},1\big]\,. \end{cases}$$

Assume there is a continuous function $u_\infty\in C^0([-1,1],\mathbb{R})$ such that $\|u_n-u_\infty\|_1\to 0$ as $n\to\infty$. Then notice that the pointwise limit of u_n is discontinuous. How can you use the knowledge of the pointwise limit to show the contradiction $\|u_n-u_\infty\|_1 \not\to 0$ as $n\to\infty$?

Exercise 10 (Optional challenging question).

Let $f:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ be continuous, and consider for an initial pair $(t_0,x_0)\in\mathbb{R}\times\mathbb{R}^d$ the initial value problem

$$\dot{x} = f(t,x)\,,\qquad x(t_0)=x_0\,. \tag{1}$$

Suppose that for a compact interval $J\subset\mathbb{R}$ with $t_0\in J$ in its interior, the Picard iterates corresponding to the initial value problem (1), given by $\big\{\lambda_n : J\to\mathbb{R}^d\big\}_{n\in\mathbb{N}_0}$, converge uniformly to a function $\lambda_\infty : J\to\mathbb{R}^d$. Show that then λ_∞ satisfies

$$\lambda_\infty(t) = x_0 + \int_{t_0}^{t} f(s,\lambda_\infty(s))\,\mathrm{d}s \quad \text{for all } \ t\in J\,,$$

and thus, λ_∞ is a solution of the initial value problem (1).

Comments on importance and difficulty of the exercises. A theme on this problem sheet is the question whether properties of a normed vector space depend on a specifically chosen norm. Exercise 6 (ii) implies that the two given norms on $\mathbb{R}^{n\times m}$ create the same topology (the same set of open sets), so both spaces are topologically the same. This holds in general for any finite-dimensional normed vector space (one can show that on each such space, any two norms are equivalent, i.e. they satisfy a relation like in Exercise 6 (ii)). The situation is different in the infinite-dimensional context, and a counterexample is given in Exercise 9. The statement of Exercise 8 is important, since it gives an easy criterion for the existence of globally defined solutions, and this is applied for linear systems $\dot{x} = Ax$ in Chapter 3. A more general criterion for the existence of global solutions will be topic of an optional challenging exercise on Problem Sheet 7). Exercises 6 and 7 concern basic questions and standard material. Exercise 8 requires an idea and a solid understanding of the global version of the Picard–Lindelöf theorem. Some comments on this exercise have also been made in Lecture 8. Exercises 9 and 10 are difficult, but the hints are very helpful. The statement in the challenging exercise is important, because it provides the final step needed to show the usefulness of the Picard iterates, see also the text after Definition 2.2. The last two exercises train you to argue solidly in analysis.