MATH50004 Differential Equations
Spring Term 2021/22
Solutions to Problem Sheet 3

Exercise 11.

We denote the right hand side by $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^{\frac{2}{3}}$.

(i) We use the separation of variables to solve the initial value problem $x(t_0) = 0$ with $t_0 \in \mathbb{R}$. We get

$$\int_0^x \frac{1}{y^{\frac{2}{3}}}\,\mathrm{d}y = \int_{t_0}^t \mathrm{d}s \quad \Longleftrightarrow \quad 3x^{\frac{1}{3}} = t - t_0 \quad \Longleftrightarrow \quad x = \frac{1}{27}(t-t_0)^3\,.$$

This solution is non-constant and goes through a zero of the right hand side f. The zero corresponds to a constant solution with value $x = 0$, and thus, we can concatenate solutions, e.g. it follows that for any $\alpha \le 0 \le \beta$, the function $\lambda_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$, given by

$$\lambda_{\alpha,\beta}(t) := \begin{cases} \frac{1}{27}(t-\alpha)^3 & : \quad t \le \alpha \\ 0 & : \quad \alpha < t < \beta \\ \frac{1}{27}(t-\beta)^3 & : \quad t \ge \beta \end{cases} ,$$

is a solution of the given initial value problem. Hence, there are infinitely many solutions of the initial value problem.

(ii) This non-uniqueness does not contradict the Theorem of Picard–Lindelöf, since the right hand side f is not Lipschitz continuous at $x = 0$. This is clear using the mean value theorem, since f is differentiable outside of $x = 0$, but the absolute value of the derivative $f'(x)$ converges to ∞ in the limit $x \to 0$.

(iii) Proposition 2.14 implies that f is locally Lipschitz continuous when restricted to $x < 0$ and $x > 0$, since f is continuously differentiable for $x \neq 0$. Hence, if a solution satisfying the initial condition $x(0) = -1$ stays negative, then it is a unique solution (see also Lemma 2.24). We replace the restriction $\alpha \le 0 \le \beta$ for the above function $\lambda_{\alpha,\beta}$ by $\alpha \le \beta$ and note that this is still a solution to the differential equation, but does not satisfy the initial condition $x(0) = 0$ from (i). The modified function $\lambda_{\alpha,\beta}$ satisfies the initial condition $x(0) = -1$ if and only if $\frac{1}{27}\alpha^3 = 1$, and that means $\alpha = 3$. Hence, the solution has a unique solution on $[0, T]$, where $0 < T \le 3$, and for $T > 3$, there are more solutions, given by $\lambda_{3,\beta}$ with $\beta > 3$.

Exercise 12.

(i) We have solved this differential equation already using separation of variables in Example 1.8, and we obtained the general solution corresponding to the initial condition $x(t_0) = x_0$, given by

$$\lambda(t, t_0, x_0) = \frac{2x_0}{2 + x_0(t_0^2 - t^2)}\,.$$

Since the differential equation is continuously differentiable, uniqueness of solutions is guaranteed due to Proposition 2.14. We now analyse for which $t \in \mathbb{R}$ the above solution $t \mapsto \lambda(t, t_0, x_0)$ exists (depending on the initial pair (t_0, x_0)), and whether it is indeed maximal. One easily calculates the maximal existence intervals on which the solution is defined, and these are given by

$$I_{max}(t_0, x_0) = \begin{cases} \left(-\sqrt{t_0^2 + \frac{2}{x_0}}, \sqrt{t_0^2 + \frac{2}{x_0}}\right) & : \quad x_0 > 0\,, \\ (-\infty, \infty) & : \quad t_0^2 + \frac{2}{x_0} < 0 \text{ or } x_0 = 0\,, \\ \left(-\infty, -\sqrt{t_0^2 + \frac{2}{x_0}}\right) & : \quad t_0^2 + \frac{2}{x_0} \ge 0\,, t_0 < 0 \text{ and } x_0 < 0\,, \\ \left(\sqrt{t_0^2 + \frac{2}{x_0}}, \infty\right) & : \quad t_0^2 + \frac{2}{x_0} \ge 0\,, t_0 > 0 \text{ and } x_0 < 0\,. \end{cases}$$

We consider the case that $I_{max}(t_0,x_0)$ is bounded above or below, since otherwise the solution exists globally. We take the time limit towards either $I_+(t_0,x_0)$ or $I_-(t_0,x_0)$ and note that

$$\left|\lim_{t\to I_+(t_0,x_0)} \lambda_{max}(t,t_0,x_0)\right| = \infty \quad \text{or} \quad \left|\lim_{t\to I_-(t_0,x_0)} \lambda_{max}(t,t_0,x_0)\right| = \infty\,, \quad \text{respectively,}$$

since the nominator of $\lambda(t,t_0,x_0)$, given by $2+x_0(t_0^2-t^2)$, converges to 0 in this case. This implies that the solution cannot be continued, and one of the cases described in Theorem 2.17 (explosion) applies.

(ii) Firstly, note that the (nonautonomous) domain (in the extended phase space) is given by $D := \mathbb{R}\times(-1,1)$. Secondly, note that the zero function solves this initial value problem, but we are looking for a non-constant solution. Such a solution may exist, since the given problem is not Lipschitz continuous at $x=0$, and indeed, we can compute a non-constant solution by using separation of variables in the following way.

Consider $x>0$ first. Then

$$\int_0^x \frac{1-\sqrt{y}}{\sqrt{y}}\,\mathrm{d}y = \int_0^t \mathrm{d}s \quad\Longleftrightarrow\quad 2\sqrt{x}-x=t \quad\Longrightarrow\quad x=\big(1-\sqrt{1-t}\big)^2 \quad \text{for all } t\in[0,1)\,.$$

Analogously, for $x<0$, one gets a solution

$$x=-\big(1-\sqrt{1+t}\big)^2 \quad \text{for all } t\in(-1,0]\,,$$

leading to a non-constant solution $\lambda:(-1,1)\to\mathbb{R}$,

$$\lambda(t)=\begin{cases} -\big(1-\sqrt{1+t}\big)^2 & : \ t\in(-1,0)\,,\\ \big(1-\sqrt{1-t}\big)^2 & : \ t\in[0,1)\,,\end{cases}$$

satisfying the initial condition $x(0)=0$ (one can verify the solution identity easily). We show that this solution is maximal. We get

$$\lim_{t\to-1}(t,\lambda(t)) = (-1,1)\in\partial D\,,$$

and

$$\lim_{t\to1}(t,\lambda(t)) = (1,1)\in\partial D\,,$$

so the solution we have found cannot be extended and is indeed maximal; its boundary behaviour matches one of the cases discussed in Theorem 2.17.

(iii) The right hand side is obviously continuously differentiable, and thus locally Lipschitz continuous in x, so the maximal solution exists due to Proposition 2.14 and Theorem 2.17. Due to Theorem 2.17, the maximal solution to the given initial value problem exists for all $t\in\mathbb{R}$ if and only if it does not explode in finite time to $\pm\infty$.

Let $\lambda_{max}: I_{max}(0,1)\to\mathbb{R}$ be the maximal solution. We get with $f(x)=2+\sin(x^2)$ that

$$\lambda_{max}(t) = 1+\int_0^t f(\lambda_{max}(s))\,\mathrm{d}s \quad \text{for all } t\in I_{max}(0,1)\,.$$

This implies that for all $t\in I_{max}(0,1)$ with $t\ge0$, we get

$$|\lambda_{max}(t)| \le 1+\int_0^t \underbrace{|f(\lambda_{max}(s))|}_{\le3}\,\mathrm{d}s \le 1+3t\,,$$

so the solution cannot explode forward in time. Analogously, for all $t\in I_{max}(0,1)$ with $t\le0$, we get

$$|\lambda_{max}(t)| \le 1+\int_t^0 \underbrace{|f(\lambda_{max}(s))|}_{\le3}\,\mathrm{d}s \le 1-3t = 1+3|t|\,,$$

so the solution cannot explode backward in time. Hence $I_{max}(0,1) = \mathbb{R}$.
We analyse the behaviour for $t \to \infty$ and get for $t \geq 0$ that

$$\lambda_{max}(t) = 1 + \int_0^t \underbrace{f(\lambda_{max}(s))}_{\geq 1} \,\mathrm{d}s \geq 1 + t \to \infty \quad \text{as } t \to \infty\,.$$

Analogously, we get for $t \leq 0$ that

$$\lambda_{max}(t) = 1 - \int_t^0 \underbrace{f(\lambda_{max}(s))}_{\geq 1} \,\mathrm{d}s \leq 1 + t \to -\infty \quad \text{as } t \to -\infty\,.$$

(iv) Since the graph of the solution λ_{max} has to lie in $D = \big\{(t,x) \in \mathbb{R}^2 : t^2 + x^2 < 1\big\}$, we immediately get $0 < t^+ \leq 1$. For all $t \in (t^-, t^+)$, we have $t^2 + \lambda_{max}(t)^2 < 1$, and hence

$$\dot{\lambda}(t) = \frac{1}{1 - \sqrt{t^2 + \lambda_{max}(t)^2}} > 0\,.$$

This means that λ_{max} is strictly monotonically increasing on (t^-, t^+) and bounded, so the limit $\lim_{t\to t^+} \lambda_{max}(t)$ exists. Due to Theorem 2.17, we get

$$\lim_{t \to t^+} \operatorname{dist}\big((t, \lambda_{max}(t)), \partial D\big) = 0$$

(note that an exploding solution is obviously not possible due to boundedness of D), which implies

$$\lim_{t\to t^+} t^2 + \lambda_{max}(t)^2 = 1\,,$$

and thus, $\lim_{t\to t^+} \lambda_{max}(t)^2 = 1 - (t^+)^2$, so

$$\lim_{t\to t^+} \lambda_{max}(t) = \sqrt{1 - (t^+)^2}\,,$$

since λ_{max} is positive for all $t \in (0, t^+)$. This positivity also means that $t^+ = 1$ is not possible, since then $\lim_{t\to t^+} \lambda_{max}(t) = 0$ in contradiction to the fact that λ_{max} is strictly monotonically increasing.

Exercise 13.
Since f is locally Lipschitz continuous, there exists a neighbourhood $U \subset D$ of (t_0, x_0) and a constant $K > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq K\|x - y\| \quad \text{for all } (t,x),(t,y) \in U\,. \tag{2.13}$$

Let $V \subset U$ be a compact set that contains (t_0, x_0) in its interior. Since continuous functions on compact sets attain a maximum, there exists an $M > 0$ such that

$$\|f(t,x)\| \leq M \quad \text{for all } (t,x) \in V\,. \tag{2.14}$$

Now choose $\tau, \delta > 0$ such that

$$W^{\tau,\delta}(t_0, x_0) = [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)} \subset V \subset D\,.$$

Since

$$W^{\frac{\tau}{2},\frac{\delta}{2}}(\tilde{t}_0, \tilde{x}_0) \subset W^{\tau,\delta}(t_0, x_0) \quad \text{for all initial pairs } (\tilde{t}_0, \tilde{x}_0) \in W^{\frac{\tau}{2},\frac{\delta}{2}}(t_0, x_0)\,,$$

the solution satisfying the initial value condition $x(\tilde{t}_0) = \tilde{x}_0$ exists on the interval $\big[\tilde{t}_0 - \tilde{h}, \tilde{t}_0 + \tilde{h}\big]$, where $\tilde{h} := \min\{\frac{\tau}{2}, \frac{1}{2K}, \frac{\delta}{2M}\}$. Importantly, $\tilde{h}$ does not depend on the specific initial pair from $W^{\frac{\tau}{2},\frac{\delta}{2}}(\tilde{t}_0, \tilde{x}_0)$, which finishes the proof.

Exercise 14.

Let $\lambda : I_{max}(0, x_0) \to \mathbb{R}^d$ be the maximal solution of this initial value problem. Define

$$\alpha(t) := \|\lambda(t)\|^2 = \langle \lambda(t), \lambda(t) \rangle \quad \text{for all } t \in I_{max}(0, x_0)\,.$$

Then we get

$$\dot\alpha(t) = 2\langle \dot\lambda(t), \lambda(t) \rangle = 2\langle f(\lambda(t)), \lambda(t) \rangle \ge 2\|\lambda(t)\|^3 = 2\alpha(t)^{\frac{3}{2}} \ge 0\,.$$

This implies that α is monotonically increasing, and due to $\alpha(0) = \|x_0\|^2 > 0$, we have $\alpha(t) > 0$ for all $t \ge 0$. Consider the differential equation

$$\dot x = x^{\frac{3}{2}}\,, \tag{A}$$

defined for $x > 0$ (and formally for all $t \in \mathbb{R}$), and let $\mu : I_{max}(0, \|x_0\|^2) \to \mathbb{R}$ be the maximal solution of this differential equation satisfying the initial condition $x(0) = \|x_0\|^2$ (note that on the domain $x > 0$, the right hand side of (A) is continuously differentiable, and due to Proposition 2.14 and Theorem 2.17, the maximal solution exists and is unique). Then $\alpha(0) = \mu(0)$ and $\dot\alpha(t) \ge 2\alpha(t)^{\frac{3}{2}} > \alpha(t)^{\frac{3}{2}}$ (here we need $\alpha(t) > 0$, as established before), so all assumptions of Exercise 4 are satisfied. Hence,

$$\alpha(t) > \mu(t) \quad \text{for all } t > 0 \text{ for which both solutions are defined.} \tag{B}$$

We use separation of variables to compute the solution μ. We get

$$\int_{\|x_0\|^2}^{x} \frac{1}{y^{\frac{3}{2}}}\,\mathrm{d}y = \int_0^t \mathrm{d}s \quad \Longleftrightarrow \quad -2\left(\frac{1}{\sqrt{x}} - \frac{1}{\|x_0\|}\right) = t \quad \Longrightarrow \quad x = \left(\frac{2}{t - \frac{2}{\|x_0\|}}\right)^2,$$

so $\mu(t) = \left(\frac{2}{t - \frac{2}{\|x_0\|}}\right)^2$ with $I_{max}(0, x_0) = \left(-\infty, \frac{2}{\|x_0\|}\right)$ (compare this with Example 2.18). So μ converges to ∞ in finite time, and due to (B), also α and λ do not exist for all $t > 0$. This finishes the proof.

Exercise 15.

Let $X = C^0([t_0 - h, t_0 + h], \mathbb{R})$ be the space of continuous real-valued functions on the interval $[t_0 - h, t_0 + h]$, and consider the mapping $P : X \to X$,

$$P(u)(t) := x_0 + \int_{t_0}^{t} f(s, u(s))\,\mathrm{d}s \quad \text{for all } t \in [t_0 - h, t_0 + h]\,.$$

Define $S := \{u \in X : \|u - x_0\|_\infty \le \delta\}$, and consider the mapping P restricted to the set S. We aim at finding a fixed point of P, which turns out to be a solution of the initial value problem due to Proposition 2.1.

Step 1. $P : S \to X$ is continuous.

Firstly, we choose $\varepsilon > 0$ and $u \in S$. Note that since f is continuous on the compact set $W^{\tau,\delta}(t_0, x_0)$, it is uniformly continuous on $W^{\tau,\delta}(t_0, x_0)$. This means that there exists a $\tilde\delta > 0$ such that

$$|f(t, x) - f(t, y)| < \frac{\varepsilon}{\tau} \quad \text{for all } t \in [t_0 - \tau, t_0 + \tau] \text{ and } x, y \in [x_0 - \delta, x_0 + \delta] \text{ with } |x - y| < \tilde\delta\,.$$

Now consider $\tilde u \in S$ such that $\|u - \tilde u\|_\infty < \tilde\delta$. Then for all $t \in [t_0 - \tau, t_0 + \tau]$, we get

$$\begin{aligned} |P(u)(t) - P(\tilde u)(t)| &= \left| \int_{t_0}^{t} (f(s, u(s)) - f(s, \tilde u(s)))\,\mathrm{d}s \right| \\ &\le \left| \int_{t_0}^{t} \underbrace{|f(s, u(s)) - f(s, \tilde u(s))|}_{< \frac{\varepsilon}{\tau}}\,\mathrm{d}s \right| < \frac{\varepsilon}{\tau}\tau = \varepsilon\,. \end{aligned}$$

Hence, $\|P(u) - P(\tilde{u})\|_\infty < \varepsilon$, and thus, P is continuous.

Step 2. $P(S) \subset S$.

Note that this is similar to Step 1 in the proof of the local version of the Picard–Lindelöf theorem. Assume that $u \in S$, and we need to show that $P(u) \in S$. For all $t \in [t_0 - h, t_0 + h]$, we have

$$|P(u)(t) - x_0| = \left|\int_{t_0}^{t} f(s, u(s))\,\mathrm{d}s\right| \le \left|\int_{t_0}^{t} |f(s, u(s))|\,\mathrm{d}s\right| \le \left|\int_{t_0}^{t} M\,\mathrm{d}s\right| \le hM \le \delta\,,$$

which implies that $P(u)(t) \in [x_0 - \delta, x_0 + \delta]$ for all $t \in [t_0 - h, t_0 + h]$, and hence, $P(S) \subset S$ is well-defined.

Step 3. S is nonempty, closed and convex.

S is clearly nonempty, and also closed with respect to $\|\cdot\|_\infty$, so it remains to show that S is convex. Let $u_1, u_2 \in S$ and $\alpha \in [0, 1]$. We show that $\alpha u_1 + (1-\alpha)u_2 \in S$. Note that for any $t \in [t_0 - \tau, t_0 + \tau]$, we have

$$\begin{aligned} |\alpha u_1(t) + (1-\alpha)u_2(t) - x_0| &= |\alpha u_1(t) + (1-\alpha)u_2(t) - \alpha x_0 - (1-\alpha)x_0| \\ &\le |\alpha(u_1(t) - x_0)| + |(1-\alpha)(u_2(t) - x_0)| \\ &\le \alpha\delta + (1-\alpha)\delta = \delta\,, \end{aligned}$$

which means that $\alpha u_1 + (1-\alpha)u_2 \in S$.

Step 4. $P(S)$ is relatively compact.

We apply the Arzelà–Ascoli theorem as suggested in the hint, which means that we need to show that $P(S)$ is equicontinuous and pointwise bounded.

We first prove that $P(S)$ is pointwise bounded. Due to $P(S) \subset S$, it suffices to show that S is pointwise bounded. Fix $t \in [t_0 - h, t_0 + h]$. Then for all $u \in S$, we have

$$|u(t)| \le \|u\|_\infty \le \|u - x_0 + x_0\|_\infty \le \|u - x_0\|_\infty + \|x_0\|_\infty \le \delta + \|x_0\|_\infty\,,$$

so $\sup_{u \in S} |u(t)| \le \delta + \|x_0\|_\infty < \infty$.

We now show that $P(S)$ is equicontinuous. We first observe that for all $u \in S$, and $t_1, t_2 \in [t_0 - h, t_0 + h]$, we have

$$|P(u)(t_2) - P(u)(t_1)| = \left|\int_{t_1}^{t_2} f(s, u(s))\,\mathrm{d}s\right| \le \left|\int_{t_1}^{t_2} \underbrace{|f(s, u(s))|}_{\le M}\,\mathrm{d}s\right| \le M|t_2 - t_1|\,.$$

Let $\varepsilon > 0$, and choose $\bar{\delta} := \frac{\varepsilon}{M} > 0$. Then for all $t_1, t_2 \in [t_0 - h, t_0 + h]$ with $|t_1 - t_2| < \delta$, we have

$$|P(u)(t_2) - P(u)(t_1)| \le M|t_2 - t_1| < M\bar{\delta} = \varepsilon \quad \text{for all } u \in S\,,$$

which proves equicontinuity.

Step 5. Application of Schauder fixed point theorem.

All assumptions of the Schauder fixed point theorem have been verified, and its application gives a fixed point of the mapping P, which is a solution to the given initial value problem.