MATH50004 Differential Equations
Spring Term 2021/22
Problem Sheet 3

Exercise 11 (Unique and non-unique solutions).

Consider the one-dimensional autonomous differential equation

$$\dot{x} = |x|^{\frac{2}{3}}\,.$$

(i) Show that there are infinitely many solutions that satisfy the initial condition $x(0) = 0$.

(ii) Explain why this non-uniqueness does not contradict the Picard–Lindelöf theorem.

(iii) Determine for which $T > 0$, there is more than one solution on the interval $[0, T]$ satisfying the initial condition $x(0) = -1$.

Exercise 12 (Maximal solutions).

(i) Consider the nonautonomous differential equation $\dot{x} = tx^2$, whose right hand side is defined for all $(t, x) \in \mathbb{R}^2$. Compute the maximal solution and the maximal existence interval $I_{max}(t_0, x_0)$ for each of the initial conditions $x(t_0) = x_0$, where $(t_0, x_0) \in \mathbb{R}^2$. Prove that the solutions you have computed are indeed maximal.

(ii) Compute a non-constant maximal solution $\lambda_{max} : (t^-, t^+) \to \mathbb{R}$ of the initial value problem

$$\dot{x} = f(x)\,, \qquad x(0) = 0\,, \qquad \text{where } f : (-1, 1) \to \mathbb{R}\,, \quad f(x) := \frac{\sqrt{|x|}}{1 - \sqrt{|x|}}\,.$$

Determine the behaviour of $\lambda_{max}(t)$ for $t \to t^-$ and $t \to t^+$.

(iii) Consider the initial value problem

$$\dot{x} = 2 + \sin(x^2)\,, \qquad x(0) = 1\,,$$

and note that the right hand side is defined for all $x \in \mathbb{R}$ (and formally for all $t \in \mathbb{R}$). Show that the maximal solution of this initial value problem exists for all $t \in \mathbb{R}$, and determine the behaviour of this solution for $t \to -\infty$ and $t \to \infty$.

(iv) Consider the nonautonomous differential equation

$$\dot{x} = \frac{1}{1 - \sqrt{t^2 + x^2}}\,,$$

the right hand side of which is defined for all $(t, x) \in D := \{(t, x) \in \mathbb{R}^2 : t^2 + x^2 < 1\}$. Let $\lambda_{max} : (t^-, t^+) \to \mathbb{R}$ be a solution satisfying the initial condition $x(0) = 0$, and assume that the interval (t^-, t^+) cannot be extended, so this solution is maximal. Show that $t^+ < 1$ and $\lim_{t \to t^+} \lambda_{max}(t) = \sqrt{1 - (t^+)^2}$.

Hint. For (iii) and (iv), it is not necessary to compute the solution of the initial value problem.

Exercise 13 (Uniformity of local existence interval in a neighbourhood of initial condition).

Consider the setting of the local version of the Picard–Lindelöf theorem, and show that for each initial condition (t_0, x_0), there exists neighbourhood U of (t_0, x_0) and an $h > 0$ such that for each $(\bar{t}_0, \bar{x}_0) \in U$, there exists a solution $\lambda_{\bar{t}_0, \bar{x}_0} : [\bar{t}_0 - h, \bar{t}_0 + h] \to \mathbb{R}^d$ to the initial value problem

$$\dot{x} = f(t, x)\,, \qquad x(\bar{t}_0) = \bar{x}_0\,.$$

Exercise 14 (Criterion for non-global existence of solutions).
Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable with

$$\langle f(x), x \rangle \geq \|x\|^3 \quad \text{for all } x \in \mathbb{R}^d .$$

Show that no solution of the initial value problem

$$\dot{x} = f(x) , \qquad x(0) = x_0 ,$$

where $x_0 \neq 0$, exists for all $t \geq 0$.

Hint. Exercise 4 is useful here.

Exercise 15 (Peano theorem – optional challenging question).
For given $\tau, \delta > 0$ and $(t_0, x_0) \in \mathbb{R}^2$, let $f : W^{\tau,\delta}(t_0, x_0) \to \mathbb{R}$ be a continuous function defined on

$$W^{\tau,\delta}(t_0, x_0) := \big\{(t, x) \in \mathbb{R}^2 : |t - t_0| \leq \tau \text{ and } |x - x_0| \leq \delta\big\} .$$

Define

$$M := \max_{(t,x)\in W^{\tau,\delta}(t_0,x_0)} |f(t, x)| \quad \text{and} \quad h := \min\big\{\tau, \tfrac{\delta}{M}\big\} .$$

Prove that there exists at least one solution $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}$ of the initial value problem

$$\dot{x} = f(t, x) , \qquad x(t_0) = x_0 .$$

Hint. There are different ways to prove this, but aim at using the following versions of the *Schauder fixed point theorem* and the *Arzelà–Ascoli theorem*, both of which you can use without proof.

Schauder fixed point theorem. Let $(X, \|\cdot\|)$ be a normed vector space and $S \subset X$ be nonempty, closed and convex, and consider a continuous mapping $P : S \to S$. If $P(S) \subset S$ is relatively compact in X (i.e. the closure of $P(S)$ is compact in X), then P has a fixed point.

Arzelà–Ascoli theorem. Consider $X = C^0([t_0 - h, t_0 + h], \mathbb{R})$, equipped with the supremum norm $\|\cdot\|_\infty$. Then a subset $T \subset X$ is relatively compact if and only if it is *equicontinuous* (i.e. for all $t \in [t_0 - h, t_0 + h]$ and $\varepsilon > 0$, there exists a $\bar{\delta} > 0$ such that for all $s \in (t - \bar{\delta}, t + \bar{\delta}) \cap [t_0 - h, t_0 + h]$ and $g \in T$, we have $|g(t) - g(s)| < \varepsilon$) and *pointwise bounded* (i.e. for all $t \in [t_0 - h, t_0 + h]$, we have $\sup_{g\in T} |g(t)| < \infty$).

Strategy of proof. Use $X = C^0([t_0 - h, t_0 + h], \mathbb{R})$, the mapping $P : S \to X$ considered in the Picard–Lindelöf theorem, restricted to the set $S := \{u \in X : \|u - x_0\|_\infty \leq \delta\}$.

Remark. The Peano theorem gives only existence of solutions, but not uniqueness. Note that the theorem does not require a Lipschitz condition and only needs continuity of the right hand side.

Comments on importance and difficulty of the exercises. Exercise 11 deals with unique and non-unique solutions in a setting similar to Example 1.7, and the arguments needed here are standard and train you on understanding the Picard–Lindelöf theorem. Exercise 12 illustrate the theoretical results on maximal solutions by means of four examples. While for (i) and (ii), all computations can be done explicitly, (iii) and (iv) focus on understanding the behaviour of solutions without knowing them explicitly. Such analyses will become more and more important in this course until the end of the term. Most differential equations cannot be solved analytically, but we still aim at understanding how the solutions behave (the existence of which has been clarified by now), and we will do so using basic tools from the so-called *Qualitative Theory of Differential Equations*. Exercise 13 is a consequence of the quantitative version of the local version of the Picard–Lindelöf theorem and is needed for the proof of Theorem 2.17 on maximal solutions. Exercise 14 gives an easily verifiable criterion for non-global existence of solutions, and the proof requires some ideas, which will be revealed on the hints to this problem sheet. The challenging exercise is a bit lengthy to execute and requires you to apply results that you may not have seen before. The result is very important, since it says that the continuity of the right hand side implies the existence of local solutions, and we do not require Lipschitz continuity for that as in the Picard–Lindelöf theorem.