MATH50004 Differential Equations
Spring Term 2021/22
Solutions to Problem Sheet 4

Exercise 16.

(i) Assume that there exists a non-monotone solution $\mu : J \to \mathbb{R}$. Then there exist $t_1 < t_2 < t_3 \in J$ such that either

$$\mu(t_1) = \mu(t_3) < \mu(t_2) \quad \text{and} \quad \mu(t_2) \geq \mu(t) \quad \text{for all } \ t \in [t_1, t_3]$$

or

$$\mu(t_1) = \mu(t_3) > \mu(t_2) \quad \text{and} \quad \mu(t_2) \leq \mu(t) \quad \text{for all } \ t \in [t_1, t_3]\,.$$

It follows that $\dot\mu(t_2) = 0$, and the solution identity implies $0 = \dot\mu(t_2) = f(\mu(t_2))$. Since a zero of the right hand side corresponds to a constant solution, the function $\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda(t) = \mu(t_2)$, is a solution and satisfies the initial condition $x(t_2) = \mu(t_2)$. But also the solution μ satisfies this initial condition, and since solutions cannot cross under the local Lipschitz condition, μ and λ must coincide. This contradicts the above assumption that μ is not monotone.

(ii) Assume to the contrary that $f(c) \neq 0$. Without less of generality, we have for the first component $f_1(c) > 0$, and the continuity of f implies

$$\lim_{t\to\infty} \dot\lambda_1(t) = \lim_{t\to\infty} f_1(\lambda(t)) = f_1\Big(\lim_{t\to\infty} \lambda(t)\Big) = f_1(c) > 0\,,$$

which implies that there exists a $T > 0$ with $\dot\lambda_1(t) \geq \frac{f_1(c)}{2}$ for all $t \geq T$. This implies the inequality

$$\lambda_1(t) = \lambda_1(T) + \int_T^t \dot\lambda_1(s)\,\mathrm{d}s \geq \lambda_1(T) + \int_T^t \frac{f_1(c)}{2}\,\mathrm{d}s = \lambda_1(T) + \frac{f_1(c)}{2}(t-T) \quad \text{for all } \ t \geq T\,,$$

the right hand side of which clearly converges to ∞ in the limit $t \to \infty$. This implies $\lim_{t\to\infty} \lambda_1(t) = \infty$, in contradiction to the assumption.

Concerning the additional questions:

Firstly, we show how to argue in (i) if f is not necessarily Lipschitz continuous. Assume that there exists a non-monotone solution $\mu : J \to \mathbb{R}$. Then there exist $t_1 < t_2 < t_3 \in J$ such that either

$$\mu(t_1) = \mu(t_3) < \mu(t_2) \quad \text{and} \quad \mu(t_2) \geq \mu(t) \quad \text{for all } \ t \in [t_1, t_3]$$

or

$$\mu(t_1) = \mu(t_3) > \mu(t_2) \quad \text{and} \quad \mu(t_2) \leq \mu(t) \quad \text{for all } \ t \in [t_1, t_3]\,.$$

We concentrate on the first case only, since the second case can be treated analogously. Define

$$\tilde t_1 := \sup\big\{t \leq t_2 : \mu(t) = \mu(t_1)\big\} \in [t_1, t_2)\,.$$

The mean value inequality implies that there exist $\tau \in (\tilde t_1, t_2)$ such that

$$f(\mu(\tau)) = \dot\mu(\tau) = \frac{\mu(\tilde t_1) - \mu(t_2)}{\tilde t_1 - t_2} > 0\,, \tag{A}$$

and we have $\mu(\tau) \in [\mu(\tilde t_1), \mu(t_2)]$ due to the intermediate value theorem. This implies $\mu(\tau) \in [\mu(t_3), \mu(t_2)]$, and thus,

$$\tilde\tau := \inf\big\{t \geq t_2 : \mu(t) = \mu(\tau)\big\} \in (t_2, t_3]\,.$$

Now for all $t \in [t_2, \tilde\tau]$, we have $\mu(t) \geq \mu(\tilde\tau)$, due to the definition of the infimum. This implies

$$f(\mu(\tilde\tau)) = \dot\mu(\tilde\tau) \leq 0\,,$$

and this contradicts (A), since $\mu(\tilde{\tau}) = \mu(\tau)$.

Secondly, the statement (ii) does not hold in the nonautonomous case. Consider the one-dimensional linear equation

$$\dot{x} = -x + \frac{1}{t} - \frac{1}{t^2}\,,$$

defined on $D := \mathbb{R}^+ \times \mathbb{R}$, and which has the general solution $\lambda(t, t_0, x_0) = \frac{1}{t} - e^{-(t-t_0)}(\frac{1}{t_0} - x_0)$ for all $(t, t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ (see also Exercise 1 or Proposition 3.10). Obviously, we get $\lim_{t\to\infty} \lambda(t, t_0, x_0) = 0$, but the linear system does not have the zero solution.

Exercise 17.

It is possible to compute the flow of this differential equation by means of separation of variables:

$$\varphi(t, x) := \frac{x}{x + (1 - x)e^{-t}}\,,$$

where the flow $\varphi(t, x)$ starting at $x < 0$ escapes in finite positive time to $-\infty$, and the flow $\varphi(t, x)$ starting at $x > 1$ escapes in finite negative time to ∞. The flow starting in $x \in [0, 1]$ exists for all times.

Note that the exercise can be solved with this explicit representation of the flow, but it is also possible to argue without this knowledge, which we will do in the following, since these arguments are applicable to any one-dimensional autonomous system. We get the following different cases depending on $x \in \mathbb{R}$.

(i) Case $x > 1$: $t \mapsto \varphi(t, x)$ is monotonically decreasing (Exercise 16 (i) and sign of right hand side) and is bounded below in the limit $t \to \infty$ (solution cannot cross the constant solution in 1, see Lemma 2.15), so must converge, and with Exercise 16 (ii), we get $\lim_{t\to\infty} \varphi(t, x) = 1$. Furthermore, in the limit $t \to -\infty$ the solution cannot converge, since this would be converge an equilibrium $x^* > 1$ (Exercise 16 (ii)), which does not exist, so it diverges: $\lim_{t\to\inf J_{max}(x)} = \infty$.

(ii) Case $x = 1$. $\varphi(t, x) = 1$ for all $t \in \mathbb{R}$.

(iii) Case $x \in (0, 1)$: with very similar arguments to (i), one shows $\varphi(t, x) \to 1$ as $t \to \infty$ and $\varphi(t, x) \to 0$ as $t \to -\infty$.

(iv) Case $x = 0$. $\varphi(t, x) = 0$ for all $t \in \mathbb{R}$.

(v) Case $x < 0$: with very similar arguments to (i), one shows $\varphi(t, x) \to -\infty$ as $t \to \sup J_{max}(x)$ and $\varphi(t, x) \to 0$ as $t \to -\infty$.

This implies

$$\begin{aligned}
&O^-(-2) = [-2, 0) \text{ and } O^+(-2) = (-\infty, -2]\,,\\
&O^-(0) = O^+(0) = \{0\}\,,\\
&O^-\big(\tfrac{1}{2}\big) = \big(0, \tfrac{1}{2}\big] \text{ and } O^+\big(\tfrac{1}{2}\big) = \big[\tfrac{1}{2}, 1\big)\,,\\
&O^-(1) = O^+(1) = \{1\}\,,\\
&O^-(2) = [2, \infty) \text{ and } O^+(2) = (1, 2]\,.
\end{aligned}$$

Exercise 18.

(i) φ_1 is obviously a flow, since it is generated by the differential equation $\dot{x} = x$. However, φ_2 is not a flow. This can be seen from

$$\varphi_2(t + s, x) = e^{(t+s)^2}x = e^{t^2}e^{2ts}e^{s^2}x = e^{2ts}e^{t^2}\varphi_2(s, x) = e^{2ts}\varphi_2(t, \varphi_2(s, x))\,,$$

and this means that the group property is violated for times $t, s \in \mathbb{R}$ with $ts \neq 0$.

(ii) We differentiate the identity $\varphi(s,\varphi(t,x)) = \varphi(s+t,x)$ with respect to s and obtain using the chain rule that

$$\frac{\partial \varphi}{\partial t}(s,\varphi(t,x)) = \frac{\mathrm{d}}{\mathrm{d}s}\varphi(s,\varphi(t,x)) = \frac{\mathrm{d}}{\mathrm{d}s}\varphi(s+t,x)) = \frac{\partial \varphi}{\partial t}(t+s,x)\,.$$

Setting $s=0$ gives

$$\frac{\partial \varphi}{\partial t}(0,\varphi(t,x)) = \frac{\partial \varphi}{\partial t}(t,x)\,,$$

and hence, the function $\varphi(\cdot,x) : \mathbb{R} \to \mathbb{R}^d$ is solution of the autonomous system

$$\dot{x} = \frac{\partial \varphi}{\partial t}(0,x)\,,$$

the right hand side of which is continuously differentiable, so we have unique existence of solutions, and the flow generated by this differential equation exists. Since the solutions $\varphi(\cdot,x)$ are defined on $\mathbb{R}$, they are maximal solutions. Moreover, we have the initial condition $\varphi(0,x) = x$ for all $x \in D$, which implies that φ is the flow of this differential equation.

Exercise 19.

Consider the general solution $\lambda : \Omega \to \mathbb{R}$ of the given differential equation. For all $(t,t_0,x_0) \in \Omega$, we get the two identities

$$\frac{\partial \lambda}{\partial t}(t,t_0,x_0) = f(t,\lambda(t,t_0,x_0)) \quad \text{and} \quad \lambda(t_0,t_0,x_0) = x_0\,. \tag{A}$$

We differentiate the first identity with respect to x_0 and obtain

$$\frac{\partial}{\partial x_0}\frac{\partial \lambda}{\partial t}(t,t_0,x_0) = \frac{\partial f}{\partial x}(t,\lambda(t,t_0,x_0)) \cdot \frac{\partial \lambda}{\partial x_0}(t,t_0,x_0)\,,$$

and we can change the order of differentiation (since λ is twice differentiable according to the hint) to obtain

$$\frac{\partial}{\partial t}\frac{\partial \lambda}{\partial x_0}(t,t_0,x_0) = \frac{\partial f}{\partial x}(t,\lambda(t,t_0,x_0)) \cdot \frac{\partial \lambda}{\partial x_0}(t,t_0,x_0)\,,$$

so $\mu(\cdot) = \frac{\partial \lambda}{\partial x_0}(\cdot,t_0,x_0) : I_{max}(t_0,x_0) \to \mathbb{R}$ satisfies the given differential equation (the variational equation). We have to still check the initial condition, which follows from differentiating the second identity in (A) with respect to x_0:

$$\mu(t_0) = \frac{\partial \lambda}{\partial x_0}(t_0,t_0,x_0) = 1\,.$$

Exercise 20.

(i) We have

$$\dot{\lambda}(t+T,t_0+T,x_0) = f\big(t+T,\lambda(t+T,t_0+T,x_0)\big) = f\big(t,\lambda(t+T,t_0+T,x_0)\big)\,,$$

and hence, $t \mapsto \lambda(t+T,t_0+T,x_0)$ is solution of (2) satisfying the initial condition $x(t_0) = x_0$. Due to uniqueness of solutions, we get

$$\lambda(t+T,t_0+T,x_0) = \lambda(t,t_0,x_0) \quad \text{for all } t \geq t_0\,.$$

To prove the second equality, note that we have

$$\dot{\lambda}(t+T,t_0,x_0) = f\big(t+T,\lambda(t+T,t_0,x_0)\big) = f\big(t,\lambda(t+T,t_0,x_0)\big)\,,$$

which means that we $t \mapsto \lambda(t+T, t_0, x_0)$ is a solution of (2) satisfying the initial condition $x(t_0) = \lambda(t_0+T, t_0, x_0)$. Again, due to uniqueness of solutions, we get

$$\lambda(t+T, t_0, x_0) = \lambda(t, t_0, \lambda(t_0+T, t_0, x_0)) \quad \text{for all } \ t \geq t_0 \,.$$

(ii) Assume that $\nu_0(t_0) \leq \nu_1(t_0)$. Firstly, $\nu_0(t_0) = \lambda(t_0, t_0, x_0) = x_0$, and hence,

$$\begin{aligned}\nu_k(t) &= \lambda(t+kT, t_0, x_0) = \lambda(t+kT, t_0, \nu_0(t_0)) \leq \lambda(t+kT, t_0, \nu_1(t_0)) \\ &= \lambda(t+kT, t_0, \lambda(t+T, t_0, x_0)) \overset{\text{(i)}}{=} \lambda(t+kT+T, t_0, x_0) = \nu_{k+1}(t)\end{aligned}$$

for all $t \geq t_0$ and $k \in \mathbb{N}_0$. The case $\nu_0(t_0) \geq \nu_1(t_0)$ can be treated analogously.

(iii) Assume without loss of generality that $t_0 = 0$. Let $x_0 = \mu(0)$. Then for the initial pair $(t_0, x_0) = (0, \mu(0))$, the general solution $\lambda(t, 0, x_0) = \mu(t)$ exists for all $t \geq t_0$, so the assumption of (i) is satisfied. We only consider the case $x_0 \leq \lambda(T, 0, x_0)$, which implies $\nu_0(0) \leq \nu_1(0)$. Due to (ii), we get $\nu_k(t) \leq \nu_{k+1}(t)$ for all $k \in \mathbb{N}_0$ and $t \geq 0$, which means that for all $t \geq 0$, the sequence $\{\nu_k(t)\}_{k\in\mathbb{N}_0}$ is monotonically increasing. Since μ is bounded, this sequence is convergent, and we define the limit function $\nu : \mathbb{R}_0^+ \to \mathbb{R}$ via the pointwise limits

$$\nu(t) = \lim_{k\to\infty} \nu_k(t) \quad \text{for all } \ t \geq 0 \,.$$

We divide the remaining proof into four steps.

Step 1. $\{\nu_k\}_{k\in\mathbb{N}_0}$ *converges uniformly on* $[0,T]$ *to* ν.
Since μ is bounded, there exists an $M > 0$ such that

$$|\nu_k(t)| = |\lambda(t+kT, 0, x_0)| = |\mu(t+KT)| \leq M \quad \text{for all } \ k \in \mathbb{N}_0 \text{ and } t \geq 0 \,.$$

The functions $\nu_k : \mathbb{R}_0^+ \to \mathbb{R}$ are also differentiable, and we get the estimate

$$|\dot{\nu}_k(t)| = |f(t, \nu_k(t))| \leq \sup_{(s,y)\in[0,T]\times[-M,M]} |f(s,y)| =: L \quad \text{for all } \ k \in \mathbb{N}_0 \text{ and } t \geq 0 \,.$$

Here, the supremum on the right hand side is finite, since the continuous function f is bounded on the compact set $[0,T] \times [-M,M]$. The fundamental theorem of calculus implies then

$$|\nu_k(t_1) - \nu_k(t_2)| = \left| \int_{t_2}^{t_1} \dot{\nu}_k(s)\,\mathrm{d}s \right| \leq L|t_1 - t_2| \quad \text{for all } \ k \in \mathbb{N}_0 \text{ and } t_1, t_2 \in [0,T] \,.$$

Let $\varepsilon > 0$. Choose a natural number $R > \frac{3LT}{\varepsilon}$ and fix $t_i := T\frac{i}{R}$ for all $i \in \{0, \dots, R\}$. For each such $i \in \{0, \dots, R\}$, the sequence $\{\nu_k(t_i)\}_{k\in\mathbb{N}_0}$ is convergent and thus a Cauchy sequence. Therefore, there exists an $N_i \in \mathbb{N}$ with

$$|\nu_n(t_i) - \nu_m(t_i)| < \frac{\varepsilon}{3} \quad \text{for all } \ n, m \geq N_i \,.$$

For an arbitrary $t \in [0,T]$, there exists an $i \in \{0, \dots, R-1\}$ with $t \in [t_i, t_{i+1}]$, and we get

$$\begin{aligned}|\nu_n(t) - \nu_m(t)| &\leq |\nu_n(t) - \nu_n(t_i)| + |\nu_n(t_i) - \nu_m(t_i)| + |\nu_m(t_i) - \nu_m(t)| \\ &\leq L|t - t_i| + \frac{\varepsilon}{3} + L|t_i - t| \leq \varepsilon \quad \text{for all } \ n, m \geq \max\{N_0, \dots, N_R\} \,.\end{aligned}$$

This implies the uniform convergence on $[0,T]$.

Step 2. The limiting function ν *is a solution of (2).*
Due to (i), we get the identity

$$\nu_k(t) = \lambda(t+kT, t_0, x_0) = \lambda(t, 0, \lambda(kT, 0, x_0)) = \lambda(t, 0, \nu_k(0)) \quad \text{for all } \ t \geq 0 \text{ and } k \in \mathbb{N}_0 \,.$$

We take the limit $k \to \infty$ and obtain

$$\nu(t) = \lambda(t, 0, \nu(0)) \quad \text{for all } t \geq 0\,,$$

where we used the continuity of the general solution. This implies that ν is a solution of (2).

Step 3. The function ν is periodic with period T. Due to (i), we have

$$\nu_k(T) = \lambda(T + kT, 0, x_0) = \nu_{k+1}(0) \quad \text{for all } k \in \mathbb{N}_0\,,$$

which implies in the limit $k \to \infty$ that $\nu(T) = \nu(0)$. Since the right-hand side is periodic with period T, this implies that the solution ν is T-periodic.

Step 4. Proof of $\lim_{t\to\infty}(\mu(t) - \nu(t)) = 0$. Let $\{t_\ell\}_{\ell\in\mathbb{N}}$ be a sequence converging to ∞, and choose a sequence $\{k_\ell\}_{\ell\in\mathbb{N}}$ such that

$$t_\ell - k_\ell T \in [0, T) \quad \text{for all } \ell \in \mathbb{N}$$

(note that this sequence is uniquely determined). Then also $k_\ell \to \infty$ as $\ell \to \infty$, and we get using Step 3 that

$$|\lambda(t_\ell, 0, x_0) - \nu(t_\ell)| = |\nu_{k_\ell}(t_\ell - k_\ell T) - \nu(t_\ell - k_\ell T)| \quad \text{for all } l \in \mathbb{N}\,.$$

Due to Step 1, the right hand side of this equality converges to 0 as $\ell \to \infty$, and thus

$$\lim_{t\to\infty} |\lambda(t, 0, x_0) - \nu(t)| = 0\,,$$

which finishes the proof.