MATH50004 Differential Equations
Spring Term 2021/22
Problem Sheet 4

Exercise 16 (Monotone and constant solutions of autonomous differential equations).
Consider an autonomous differential equation

$$\dot{x} = f(x)\,, \tag{1}$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous on an open set $D \subset \mathbb{R}^d$. Show that

(i) every solution of (1) is monotone in the one-dimensional case $d = 1$,

(ii) if there exists a solution $\lambda : I \to \mathbb{R}^d$ on an interval I that is unbounded above, and we have $\lim_{t\to\infty} \lambda(t) = c \in D$, then $\mu(t) = c$ for all $t \in \mathbb{R}$ is also a solution of (1).

Does (i) also hold when f is only continuous and not necessarily locally Lipschitz continuous? Does (ii) also hold when (1) is nonautonomous?

Remark. This proves Proposition 2.27, which says that there do not exist periodic orbits in one-dimensional autonomous differential equations.

Exercise 17 (Half-orbits).
Compute for the one-dimensional autonomous differential equation

$$\dot{x} = x - x^2$$

the half-orbits $O^-(-2), O^-(0), O^-(\frac{1}{2}), O^-(1), O^-(2), O^+(-2), O^+(0), O^+(\frac{1}{2}), O^+(1), O^+(2)$.

Exercise 18 (Flows).
Consider an open set $D \subset \mathbb{R}^d$, and let $\varphi : \mathbb{R} \times D \to D$ an (abstract) flow, i.e. the function φ satisfies

$$\varphi(0, x) = x \quad \text{for all } x \in D\,,$$
$$\varphi(t, \varphi(s, x)) = \varphi(t + s, x) \quad \text{for all } t, s \in \mathbb{R} \text{ and } x \in D\,.$$

(i) Are the functions $\varphi_1, \varphi_2 : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\varphi_1(t, x) := e^t x \quad \text{and} \quad \varphi_2(t, x) := e^{t^2} x\,,$$

flows generated by an autonomous differential equation?

(ii) Show that if φ is two-times continuously differentiable, then there exists an autonomous differential equation with domain D that generates the given (abstract) flow φ.

Hint. Differentiate the identity $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ appropriately.

Exercise 19 (Variational equation).
Consider a continuously differentiable function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and let $\lambda : \Omega \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the general solution of the one-dimensional differential equation

$$\dot{x} = f(t, x)\,.$$

Show that for a fixed $(t_0, x_0) \in \mathbb{R}^2$, the function

$$\mu(t) := \frac{\partial \lambda}{\partial x_0}(t, t_0, x_0)\,, \qquad \mu : I_{max}(t_0, x_0) \to \mathbb{R}\,,$$

is the maximal solution of the linear initial value problem

$$\dot{y} = \frac{\partial f}{\partial x}\big(t, \lambda(t, t_0, x_0)\big) \cdot y\,, \qquad y(t_0) = 1\,. \tag{2}$$

Hint. You can use without proof that the general solution λ is two times continuously differentiable.

Remark. The linear differential equation in (2) is called *variational equation* along the solution $t \mapsto \lambda(t, t_0, x_0)$. This differential equation describes what happens in first order when the initial value x_0 is perturbed. Its analysis can often reveal that a solution is *stable*, in the sense that perturbations of the initial values do not matter for the long-term behaviour of the solution, or it can show the opposite (we will treat such questions in Chapter 4). This motivates that studying linear systems (like we do in Chapter 3) is of utmost importance for the understanding of nonlinear systems.

Exercise 20 (Optional challenging question).

Consider the differential equation

$$\dot{x} = f(t, x)\,, \tag{3}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and locally Lipschitz continuous with respect to x. We assume that f is periodic in t, i.e. for some $T > 0$, we have

$$f(t, x) = f(t + T, x) \quad \text{for all } (t, x) \in \mathbb{R}^2\,.$$

Prove the following three statements.

(i) Suppose that for an initial pair $(t_0, x_0) \in \mathbb{R}^2$, the general solution $\lambda(t, t_0, x_0)$ exists for all $t \geq t_0$. Then we have a translation invariance of the form

$$\begin{aligned} \lambda(t + T, t_0 + T, x_0) &= \lambda(t, t_0, x_0)\,, \\ \lambda(t + T, t_0, x_0) &= \lambda(t, t_0, \lambda(t_0 + T, t_0, x_0)) \end{aligned}$$

for all $t \geq t_0$.

(ii) The sequence of functions $\nu_k : [t_0, \infty) \to \mathbb{R}$, given by $\nu_k(t) := \lambda(t + kT, t_0, x_0)$ is monotone in the sense

$$\begin{aligned} \nu_0(t_0) \leq \nu_1(t_0) &\Longrightarrow \nu_k(t) \leq \nu_{k+1}(t)\,, \\ \nu_0(t_0) \geq \nu_1(t_0) &\Longrightarrow \nu_k(t) \geq \nu_{k+1}(t) \end{aligned}$$

for all $k \in \mathbb{N}_0$ and $t \geq t_0$.

(iii) If $\mu : \mathbb{R}_0^+ \to \mathbb{R}$ is a bounded solution of (3), then there exists a T-periodic solution $\nu : [t_0, \infty) \to \mathbb{R}$ of (3) such that

$$\lim_{t \to \infty} (\mu(t) - \nu(t)) = 0\,.$$

Hint. Use without proof that the general solution $\lambda(t, t_0, x_0)$ depends continuously on x_0.

Comments on importance and difficulty of the exercises. Both statements of Exercise 16 are very important and will be used later frequently in this course, and the solutions are not too complicated. However, the rigorous justification of one of the additional questions seems to be surprisingly complicated to write down (if you have a better solution that the one I found, do let me know). Exercise 17 can be solved in two different ways. Firstly, the flow can be computed explicitly, from which the representations of the different half-orbits follow quickly. The second approach uses Exercise 16 and is applicable to similar problems, even when a solution cannot be computed explicitly. Exercise 18 deals with flows in a more abstract setting; the solution is quick to write down, but requires the right strategy. Exercise 19 is extremely important, since it is fundamental for perturbation analysis of differential equations. Also here the solution is not too complicated, but it may not be straightforward to see this quickly. The challenging Exercise 20 discusses the occurrence of a periodic solution for a periodic differential equation, under a quite general setting (essentially only the existence of a bounded solution is assumed). It is easy to see that such a period solution does not always exist for periodic differential equation (can you find an example?). Periodic solutions will play an important role towards the end of the course, but we will discuss this only in the context of the autonomous differential equations. The solution to the challenging exercise is quite lengthy and non-trivial, so I have divided it into several steps to guide you.