MATH50004 Differential Equations
Spring Term 2021/22
Solutions to Problem Sheet 5

Exercise 21.

(i) We have $T^{-1}(B+C)T = T^{-1}BT + T^{-1}CT$ and $(T^{-1}BT)^k = T^{-1}B^kT$. Then for any $n \in \mathbb{N}$, we get

$$T^{-1}\left(\sum_{k=0}^{n} \frac{B^k}{k!}\right) T = \sum_{k=0}^{n} \frac{(T^{-1}BT)^k}{k!} = \sum_{k=0}^{n} \frac{C^k}{k!}.$$

The statement follows in the limit $n \to \infty$.

(ii) Note that in the proof of Theorem 3.3, we showed that $e^{B(t+s)} = e^{Bt}e^{Bs}$ for any $t, s \in \mathbb{R}$. We take $t = 1$ and $s = -1$ and get $\mathrm{Id}_d = e^0 = e^B e^{-B}$, which implies the claim.

(iii) If $BC = CB$, then it follows that $Be^{Ct} = e^{Ct}B$ and $Ce^{Bt} = e^{Bt}C$ for all $t \in \mathbb{R}$. We consider the function $g(t) := e^{(B+C)t}e^{-Ct}e^{-Bt}$ for all $t \in \mathbb{R}$. Due to Theorem 3.3, we get from the product rule for derivatives

$$\begin{aligned}\dot{g}(t) &= (B+C)e^{(B+C)t}e^{-Ct}e^{-Bt} + e^{(B+C)t}(-C)e^{-Ct}e^{-Bt} + e^{(B+C)t}e^{-Ct}(-B)e^{-Bt} \\ &= (B+C)g(t) - Cg(t) - Bg(t) = 0 \in \mathbb{R}^{d\times d}\,.\end{aligned}$$

Since $\dot{g}(t) = 0$ for all $t \in \mathbb{R}$, it follows that g is a constant matrix in $\mathbb{R}^{d\times d}$. For $t = 0$, we get $g(0) = \mathrm{Id}_d$, so $g(t) = \mathrm{Id}_d$ for all $t \in \mathbb{R}$, which implies the statement using (ii).

(iv) follows directly from $B^k = \mathrm{diag}(B_1^k, \dots, B_p^k)$ for any $k \in \mathbb{N}$.

Exercise 22.

We have to prove that every trajectory of $\dot{x} = Jx$ is mapped onto a trajectory of $\dot{x} = Ax$ via the invertible linear mapping T. More precisely, we show that

$$TO_J(x) = T\{e^{Jt}x : t \in \mathbb{R}\} = O_A(Tx) \quad \text{for all } x \in \mathbb{R}^d\,,$$

where O_J and O_A denote the orbits of the differential equations $\dot{x} = Jx$ and $\dot{x} = Ax$, respectively. This follows from

$$\begin{aligned}TO_J(x) &= T\{e^{Jt}x : t \in \mathbb{R}\} = \{Te^{Jt}x : t \in \mathbb{R}\} \\ &= \{Te^{Jt}T^{-1}Tx : t \in \mathbb{R}\} = \{e^{At}Tx : t \in \mathbb{R}\} = O_A(Tx)\,,\end{aligned}$$

where we have used Exercise 21 (i).

Exercise 23.

(i) To compute e^{At} for the first matrix, we note that we have the decomposition

$$A = \underbrace{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}_{=:D} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=:P},$$

and the diagonal matrix P commutes with the nilpotent matrix D, i.e. $DP = PD$. Using Exercise 21 (iii) we get $e^{Jt} = e^{Dt}e^{Pt}$ for all $t \in \mathbb{R}$. We compute

$$e^{Pt} = \sum_{k=0}^{\infty} \frac{(Pt)^k}{k!} = \mathrm{Id}_2 + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\,.$$

Since $e^{Dt} = e^{at}\,\mathrm{Id}_2$, we get

$$e^{At} = \begin{pmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{pmatrix} \quad \text{for all } t \in \mathbb{R}\,.$$

The computation of e^{At} for the second matrix A follows a similar strategy, and we use the decomposition

$$A = \underbrace{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}_{=:D} + \underbrace{\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}}_{=:P}\,.$$

Also in this situation, the matrices D and P commute, but the matrix P is not nilpotent. However, the computation of e^{Pt} is not too complicated. Consider the matrix

$$\tilde{P} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,$$

and note that $P = b\tilde{P}$. We get

$$\tilde{P}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \tilde{P}^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -P\,, \quad \tilde{P}^4 = \mathrm{Id}_2 \quad \text{and} \quad \tilde{P}^5 = \tilde{P}\,.$$

Using the power series expansion of sine and cosine, we arrive at

$$e^{Pt} = \sum_{k=0}^{\infty} \frac{(bt\tilde{P})^k}{k!} = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix} \quad \text{for all } t \in \mathbb{R}\,,$$

and this implies that

$$e^{At} = e^{at} \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix} \quad \text{for all } t \in \mathbb{R}\,. \tag{1}$$

(ii) The characteristic polynomial is given by

$$\chi(\lambda) = \det(A - \lambda\,\mathrm{Id}) = (1-\lambda)(-1-\lambda) - \alpha = \lambda^2 - 1 - \alpha \quad \text{for all } \lambda \in \mathbb{C}\,.$$

We first consider $\alpha = 0$. We obtain the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, and one computes the corresponding eigenspaces

$$\ker(A - \lambda_1\,\mathrm{Id}) = \ker \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and

$$\ker(A - \lambda_2\,\mathrm{Id}) = \ker \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

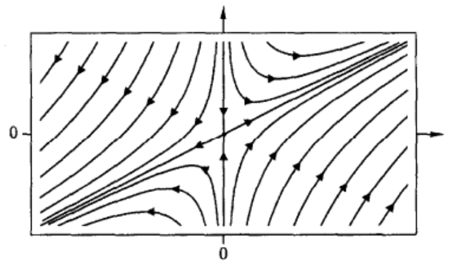
Using the transformation T (consisting of eigenvectors in its columns) and its inverse T^{-1}, given by

$$T := \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad T^{-1} := \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}\,,$$

the matrix exponential function $t \mapsto e^{At}$ reads as

$$e^{At} = Te^{Jt}T^{-1} = \begin{pmatrix} e^t & 0 \\ \frac{1}{2}(e^t - e^{-t}) & e^{-t} \end{pmatrix} \quad \text{for all } t \in \mathbb{R}\,,$$

where we used that $e^{Jt} = \mathrm{diag}(e^t, e^{-t})$. Note that the phase portrait for this system is obtained from the phase portrait in Jordan normal form (which is the saddle discussed in Section 3, I.(C1) of Chapter 3) using the linear transformation $x \mapsto Tx$ (as shown in Exercise 22), see the figure below.



We now compute the Lyapunov exponents of solutions $t \mapsto \varphi(t,x)$ for all initial conditions $x \in \mathbb{R}^2 \setminus \{0\}$. Fix such an initial condition $x = (x_1, x_2) \in \mathbb{R}^2$, where $x \neq 0$. Note that

$$\begin{aligned}\|\varphi(t,x)\| = \left\|e^{At}x\right\| &= \sqrt{x_1^2 e^{2t} + \frac{x_1^2}{4}(e^{2t} + e^{-2t} - 2) + x_2^2 e^{-2t} + x_1 x_2 (e^t - e^{-t})e^{-t}} \\ &= e^t \sqrt{x_1^2 + \frac{x_1^2}{4}(1 + e^{-4t} - 2e^{-2t}) + x_2^2 e^{-4t} + x_1 x_2 (e^t - e^{-t}) e^{-3t}}\end{aligned}$$

From this representation, it follows that if $x_1 \neq 0$, then there exists a $K = K(x_1, x_2) > 0$ such that

$$\frac{1}{K} e^t \leq \|\varphi(t,x)\| \leq K e^t \quad \text{for all } t \geq 0\,,$$

and we get

$$\sigma_{Lyap}(\varphi(\cdot,x)) = \lim_{t\to\infty} \ln \frac{\|\varphi(t,x)\|}{t} = 1\,,$$

since the Lyapunov exponent of Ce^t is equal to 1 for all $C > 0$. Now consider the case $x_1 = 0$, which implies $x_2 \neq 0$. Then $\|\varphi(t,x)\| = |x_2| e^{-t}$, and it follows that

$$\sigma_{Lyap}(\varphi(\cdot,x)) = \lim_{t\to\infty} \ln \frac{\|\varphi(t,x)\|}{t} = -1\,,$$

so we get

$$\sigma_{Lyap}(\varphi(\cdot,x)) = \begin{cases} -1 & : \quad x \in \operatorname{span}\begin{pmatrix}0\\1\end{pmatrix} \\ 1 & : \quad \text{otherwise}\end{cases}$$

Note that the Lyapounov exponent is -1 when the solution starts in the eigenspace with eigenvalue -1, and 1 otherwise. This is due to the fact that those solutions with $x_1 \neq 0$ converge in forward time to the eigenspace with eigenvalue 1.

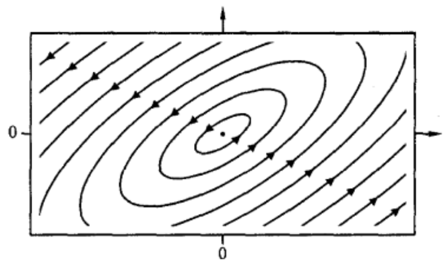
We now consider the case $\alpha = -2$ and we obtain the eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$, and one computes the eigenvectors $v_1 = \begin{pmatrix}2\\1+i\end{pmatrix}$ and $v_2 = \begin{pmatrix}2\\1-i\end{pmatrix}$. Using the real part and imaginary part of the eigenvector v_1, as discussed in the lectures, we get the same transformation matrix T as in the case $\alpha = 0$ (this is pure coincidence and will not be true for other values of α). We then calculate the Jordan normal form

$$J = T^{-1}AT = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix},$$

and thus,

$$e^{Jt} = \begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix} \quad \text{and} \quad e^{At} = Te^{Jt}T^{-1} = \begin{pmatrix}\cos(t)+\sin(t) & -2\sin(t)\\ \sin(t) & \cos(t)-\sin(t)\end{pmatrix}.$$

Using the linear transformation $x \mapsto Tx$ applied to the phase portrait of the system in Jordan normal form (see Section 3 of Chapter 3), the phase portrait of the given linear system for $\alpha = -2$ is given as in the figure below. Note that, in comparison to the case $\alpha = 0$, it is not so easy to see what this transformation does exactly to the circles around the origin. However, one can show that the resulting images, as they are linear transformations of circles, are given by ellipses. This, and the exact location of the corresponding half axes of the ellipse, follows from the polar decomposition $T = US$ into a orthogonal matrix U and a symmetric matrix S, but this is a side remark and not something I expect you to understand.



It follows from e^{At} that for all solutions $t \mapsto \varphi(t,x)$ for some $0 \neq x \in \mathbb{R}^2$, we get the existence of a $K > 0$ such that

$$\frac{1}{K} \leq \|\varphi(t,x)\| \leq K \quad \text{for all } t \geq 0\,.$$

This implies that all Lyapunov exponents are 0.

Exercise 24.

The given system $\dot{x} = Ax$ with $A \in \mathbb{R}^{3\times 3}$ is already given in Jordan normal form, and we obtain immediately that the flow is given by

$$\varphi(t,x) = e^{At}x = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt}\cos(ct) & e^{bt}\sin(ct) \\ 0 & -e^{bt}\sin(ct) & e^{bt}\cos(ct) \end{pmatrix} x\,.$$

This implies that the given statements (i)–(v) hold for the following parameter settings:
(i): $a, b < 0$, (ii): $a, b \leq 0$, (iii): $a, b > 0$, (iv): $a = b = 0$, (v): $a, b \neq 0$.
Note that c does not have an influence on all of these properties, since it concerns the speed and direction of the rotation.

Exercise 25.

Step 1. Model for a simpler situation.
We first try to understand how to model the changes in salt concentration using a simpler situation with only one tank K with 1000 litres, and we assume that at time $t_0 = 0$, per minute b litres of saline solution flow out and b litres of pure water flow in. Fix a time $t > 0$, and we are now interested to find the salt concentration $s(t)$, given an initial salt concentration $s(0)$ at time $t_0 = 0$ (note that we assume that the units for time are minutes).

We discretise the time interval $[0,t]$ into $n \in \mathbb{N}$ subintervals of length $\Delta^{(n)} := \frac{t}{n}$, and after analysing this, we consider the limit $n \to \infty$. For each discretisation depth $n \in \mathbb{N}$, we get approximations

$$s(0) = \tilde{s}(0)\,, \quad \tilde{s}\big(\Delta^{(n)}\big)\,, \quad \tilde{s}\big(2\Delta^{(n)}\big)\,, \ldots\,, \quad \tilde{s}\big(n\Delta^{(n)}\big) = \tilde{s}(t) \approx s(t)$$

for the salt concentrations at the n times $\ell\Delta^{(n)}$, where $\ell \in \{1, \ldots, n\}$.

Within the time interval $\Delta^{(n)}$, we observe an the outflow of saline solution of $b\frac{t}{n}$ litres, so the salt concentration decreases as follows:

$$\tilde{s}\big((\ell+1)\Delta^{(n)}\big) = \tilde{s}\big(\ell\Delta^{(n)}\big) - \frac{b\frac{t}{n}}{1000} s\big(\ell\Delta^{(n)}\big) \quad \text{for all } \ell \in \{0, \ldots, n-1\}\,.$$

With $\alpha := -\frac{b}{1000}$, we get

$$\tilde{s}\big((\ell+1)\Delta^{(n)}\big) = \Big(1 + \frac{\alpha t}{n}\Big)\tilde{s}\big(\ell\Delta^{(n)}\big) \quad \text{for all } \ell \in \{0, \dots, n-1\}\,.$$

By induction, it follows that

$$\tilde{s}\big((\ell+1)\Delta^{(n)}\big) = \Big(1 + \frac{\alpha t}{n}\Big)^{\ell+1}\tilde{s}(0) \quad \text{for all } \ell \in \{0, \dots, n-1\}\,,$$

and in particular

$$\tilde{s}(t) = \Big(1 + \frac{\alpha t}{n}\Big)^{n}\tilde{s}(0)\,.$$

In the limit $n \to \infty$, we get

$$s(t) = e^{\alpha t} s(0)\,,$$

so the salt concentration satisfies the differential equation

$$\dot{s} = \alpha s\,,$$

where $\alpha = -\frac{b}{1000}$ is the proportion of saline solution leaving the tank in one time unit.

Step 2. The model for $(s_1(t), s_2(t))$*.* Using the model established in Step 1, one gets the differential equation

$$\begin{aligned}\dot{s}_1 &= \frac{-80}{1000}s_1 + \frac{20}{1000}s_2\,,\\ \dot{s}_2 &= \frac{80}{1000}s_1 - \frac{80}{1000}s_2\,,\end{aligned}$$

which we can write as

$$\dot{s} = As\,, \quad \text{where } A := \begin{pmatrix} -\frac{2}{25} & \frac{1}{50} \\ \frac{2}{25} & -\frac{2}{25} \end{pmatrix}, \quad \text{and we consider the initial condition } s(0) = \begin{pmatrix} 0.05 \\ 0.02 \end{pmatrix}.$$

Step 3. Answer to (i).
One can show that A has the eigenvalues $\lambda_1 = \frac{-1}{25}$ and $\lambda_2 = \frac{-3}{25}$, which immediately implies using Proposition 3.9 that $\lim_{t\to\infty}(s_1(t), s_2(t)) = (0, 0)$. However, it is useful to calculate the flow to answer the second question. Note that the eigenvectors to the above eigenvalues are given by $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, respectively. We thus get

$$e^{At} = Te^{Jt}T^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} e^{-\frac{t}{25}} & 0 \\ 0 & e^{-\frac{3t}{25}} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-\frac{t}{25}} + \frac{1}{2}e^{-\frac{3t}{25}} & \frac{1}{4}e^{-\frac{t}{25}} - \frac{1}{4}e^{-\frac{3t}{25}} \\ e^{-\frac{t}{25}} - e^{-\frac{3t}{25}} & \frac{1}{2}e^{-\frac{t}{25}} + \frac{1}{2}e^{-\frac{3t}{25}} \end{pmatrix},$$

which implies for all $t \geq 0$ that

$$\begin{aligned}s_1(t) &= 0.03e^{-\frac{t}{25}} + 0.02e^{-\frac{3t}{25}}\,,\\ s_2(t) &= 0.06e^{-\frac{t}{25}} - 0.04e^{-\frac{3t}{25}}\,,\end{aligned}$$

and we get

$$\frac{s_1(t)}{s_2(t)} = \frac{3e^{-\frac{t}{25}} + 2e^{-\frac{3t}{25}}}{6e^{-\frac{t}{25}} - 4e^{-\frac{3t}{25}}} = \frac{3 - 2e^{-\frac{t}{25}}}{6 - 4e^{-\frac{t}{25}}} \to \frac{1}{2} \quad \text{as } t \to \infty\,.$$

Note that this is a clear analytical result, but can you link this to the eigenvectors? Also think about whether this asymptotic proportion depends on the initial condition (without doing the calculations)?

Step 4. Answer to (ii).
If instead of water, 60 litres of saline solution with 10% salt concentration will be pumped into tank K_1, then this will be modelled by the inhomogeneous system

$$\dot{s} = As + \begin{pmatrix} 0.006 \\ 0 \end{pmatrix}, \tag{B}$$

with the initial condition $(s_1(0), s_2(0)) = (0.05, 0.02)$. One can show that any solution to this inhomogeneous linear system can be represented by a particular solution of the inhomogeneous solution and a solution of the homogeneous system (which was used above in (i)), see also the proof of Proposition 3.10. Since all solutions of the homogeneous problem converge to $(0, 0)$, all solutions of the inhomogeneous system converge to the particular solution forward in time. The problem setting suggests that for both tanks, we get a salt concentration of 10%, which would mean that

$$\mu^*(t) = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} \quad \text{for all } t \geq 0$$

is a reasonable candidate for such a solution, and one verifies that this is indeed a solution of the inhomogeneous system (B). This means in particular (using the arguments established above) that

$$\lim_{t\to\infty} \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$$

for the solution of the inhomogeneous equation (B) satisfying the initial condition under consideration $(s_1(0), s_2(0)) = (0.05, 0.02)$ (and also any other initial condition).