MATH50004 Differential Equations
Spring Term 2021/22
Solutions to Problem Sheet 6

Exercise 26.

(i) The right hand side is zero if and only if

$$\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \Longleftrightarrow \quad x = \begin{pmatrix} -2 \\ 0 \end{pmatrix},$$

so $t \mapsto \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ is a constant solution of the inhomogeneous system.

(ii) We first compute the flow of the homogenous system

$$\dot{x} = \underbrace{\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}}_{=:A} x\,.$$

A straightforward computation yields that A has the double eigenvalue 1, and one finds that the eigenspace to this eigenvalue is one-dimensional, spanned by the eigenvector $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We make the ansatz $(A - \mathrm{Id}_2)w = v$ to find a generalised eigenvector w, and this ansatz leads to $w = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. With the transformation matrix T and the inverse T^{-1}, given by

$$T = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},$$

we arrive at the Jordan normal form

$$J = T^{-1}AT = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad e^{Jt} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix},$$

and we get

$$e^{At} = Te^{Jt}T^{-1} = \begin{pmatrix} e^t - 2te^t & 4te^t \\ -te^t & e^t + 2te^t \end{pmatrix}.$$

Now it is possible to use the variations of constant formula from Proposition 3.10 to compute the flow of this differential equation. However, a faster way to get there is by realising that the sum of a solution to the homogeneous system and a solution to the inhomogeneous system is a solution to the inhomogeneous system. This was something we have verified in the proof of Proposition. Using this insight, we get

$$\varphi(t, x_1, x_2) = \underbrace{\begin{pmatrix} -2 \\ 0 \end{pmatrix}}_{\text{sol. to inh. system}} + \underbrace{e^{At}\begin{pmatrix} x_1 + 2 \\ x_2 \end{pmatrix}}_{\text{sol. to hom. system}} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} e^t - 2te^t & 4te^t \\ -te^t & e^t + 2te^t \end{pmatrix}\begin{pmatrix} x_1 + 2 \\ x_2 \end{pmatrix}.$$

To see this, note that this function is a solution to the inhomogeneous system (as noted above), and in addition to that, the initial value condition (as required for the representation of the flow) is satisfied.

Exercise 27.

(i) x^* is attractive, which means that there exists an $\eta > 0$ such that for all $x \in (x^* - \eta, x^* + \eta)$, we have $\lim_{t\to\infty} \varphi(t, x) = x^*$. Here φ denotes the flow of $\dot{x} = f(x)$. To show that x^* is stable, let $\varepsilon > 0$. We define $\delta := \min\left\{\varepsilon, \frac{\eta}{2}\right\}$. Fix an $x \in (x^* - \delta, x^* + \delta)$. Then we have

$$\lim_{t\to\infty} \varphi(t, x) = x^*,$$

since $\delta < \eta$ and x^* is attractive. Since $t \mapsto \varphi(t,x)$ is monotone, we get $|\varphi(t,x) - x^*| < \delta \leq \varepsilon$. Hence, x^* is stable.

(ii) Let φ denote the flow generated by $\dot{x} = f(x)$. We prove the following: If n is odd and $f^{(n)}(x^*) < 0$, then x^* is attractive (and also stable due to (i)). If n is odd and $f^{(n)}(x^*) > 0$, then x^* is unstable. To prove this, we analyse the sign of the right hand side in a neighbourhood of x^*.

Case 1. n odd and $f^{(n)}(x^*) > 0$.
There exists a $\eta > 0$ such that $f(x) > 0$ for all $x \in (x^*, x^* + \eta)$, and $f(x) < 0$ for all $x \in (x^* - \eta, x^*)$. To show that x^* is unstable, let $\varepsilon := \frac{\eta}{2}$, and choose $\delta > 0$ arbitrarily. Without loss of generality, we assume that $\delta \leq \frac{\eta}{2}$. Then $t \mapsto \varphi(t, \frac{\delta}{2})$ is monotonically increasing (sign of f and Exercise 16 (i)) and either converges to a real number c or diverges to ∞. If it converges to a real number, Exercise 16 (ii) implies that c is a zero of the right hand side. Since there is no zero in $(x^*, x^* + \eta)$, both cases imply that there exists a $t \geq 0$ such that $\varphi(t, \frac{\delta}{2}) > x^* + \varepsilon$, which implies instability.

Case 2. n odd and $f^{(n)}(x^*) < 0$.
There exists a $\eta > 0$ such that $f(x) < 0$ for all $x \in (x^*, x^* + \eta)$, and $f(x) > 0$ for all $x \in (x^* - \eta, x^*)$. Due to the sign of f and Exercise 16 (i), any solution starting in $B_\eta(x^*)$ is monotone in the direction of x^*. Since $t \mapsto x^*$ is constant solution, those monotone solutions stay on the same side of x^* for all positive times. They do converge for this reason, but the limit must be a fixed point (Exercise 16 (ii)). Since the only fixed point in $B_\eta(x^*)$ is given by x^*, they converge to x^* and thus, x^* is attractive and stable.

Case 3. n even and $f^{(n)}(x^*) > 0$.
There exists a $\eta > 0$ such that $f(x) > 0$ for all $x \in (x^* - \eta, x^* + \eta) \setminus \{x^*\}$. Then a similar analysis to Case 1 shows that x^* is unstable.

Case 4. n even and $f^{(n)}(x^*) < 0$.
There exists a $\eta > 0$ such that $f(x) < 0$ for all $x \in (x^* - \eta, x^* + \eta) \setminus \{x^*\}$. Then a similar analysis to Case 1 shows that x^* is unstable.

Note that the sign of f close to x^* follows from the Taylor expansion

$$f(x) = \frac{f^{(n)}(x^*)}{n!}(x - x^*)^n + R(x, x^*)\,, \quad \text{where } \lim_{x \to x^*} \frac{R(x, x^*)}{|x - x^*|^n} = 0\,.$$

and you may want to prove the above statements as an exercise.

Exercise 28.

We use polar coordinates from Quiz 1 and get the system in polar coordinates, given by

$$\begin{aligned} \dot{r} &= -r - r^3\big(\cos^4(\phi) + \sin^4(\phi)\big)\,, \\ \dot{\phi} &= -1 + r^2 \cos(\phi)\sin(\phi)\big(\cos^2(\phi) - \sin^2(\phi)\big)\,. \end{aligned}$$

Consider an initial condition $(x_0, y_0) \in \mathbb{R}^2 \setminus \{0\}$ in Euclidean coordinates. We aim at proving that $\varphi(t, x_0, y_0)$ converges to the trivial equilibrium as $t \to \infty$. Choose corresponding initial conditions $r_0 := \sqrt{x_0^2 + y_0^2}$ and ϕ_0 in polar coordinates such that $x_0 = r_0 \cos(\phi_0)$ and $y_0 = r_0 \sin(\phi_0)$, and let $\mu : I_{max}(r_0, \phi_0) \to \mathbb{R}^2$ be the maximal solution of the initial value problem in polar coordinates (which exists, since the right hand side of the polar coordinate system is continuously differentiable). Using the hint on Quiz 1, we get that

$$\nu(t) = \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix} := \begin{pmatrix} \mu_1(t)\cos(\mu_2(t)) \\ \mu_1(t)\sin(\mu_2(t)) \end{pmatrix} \quad \text{for all } t \in I_{max}(r_0, \phi_0)$$

is a solution to the system in Euclidean coordinates satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. Clearly,

$$\|\nu(t)\| = \mu_1(t) \quad \text{for all } t \in I_{max}(r_0, \phi_0)\,,$$

which is monotonically decreasing, since the right hand side of $\dot{r}$ is negative. This implies that $\sup I_{max}(r_0, \phi_0) = \infty$ (using Theorem 2.17: the system is globally defined, and the solution does not explode due to the fact that the radial component decreases monotonically). Note that the right hand side of $\dot{r}$ satisfies the inequality

$$-r - r^3\big(\cos^4(\phi) + \sin^4(\phi)\big) < -r \quad \text{for all } r > 0\,.$$

Using Exercise 4, we get that $\|\nu(t)\|$ decreases faster than the solution of $\dot{r} = -r$ for the initial condition $r(0) = r_0 = \|\nu(0)\|$. Since this solution converges to 0 as $t \to \infty$, we get

$$\lim_{t\to\infty} \|\nu(t)\| = 0\,,$$

which finishes the proof.

Exercise 29.

Let J be the real Jordan normal form of the matrix A, i.e. there exists a $T \in \mathbb{R}^{d\times d}$ such that $J = T^{-1}AT$. Using Proposition 3.4 (i), we get

$$\varphi(t, x) = e^{At}x = Te^{Jt}T^{-1}x \tag{A}$$

for the flow of $\dot{x} = Ax$.

($\Rightarrow$) We show that if there exists an eigenvalue ρ of A with $\operatorname{Re}\rho \geq 0$, then $x^* = 0$ is not exponentially stable. It follows from Proposition 3.8 that $t \mapsto e^{J_j t}$, for a Jordan block corresponding to the eigenvalue ρ, does not converge to 0 as $t \to \infty$. Hence, (A) implies that the matrix e^{At} does not converge to 0 as $t \to \infty$, say the ℓ-th column does not converge to 0. Now we have

$$\varphi\big(t, \underbrace{\tfrac{\delta}{2}e_\ell}_{\in B_\delta(0)}\big) = \tfrac{\delta}{2}\varphi\big(t, e_\ell\big) = \tfrac{\delta}{2}e^{At}e_\ell \not\to 0 \quad \text{as } t \to \infty\,.$$

Hence, x^* is not attractive and thus not exponentially stable.

($\Leftarrow$) It follows from Proposition 3.9 that if all eigenvalues have negative real part, then there exists a $K > 0$ such that

$$\big\|e^{At}\big\| \leq Ke^{\gamma t} \quad \text{for all } t \geq 0\,,$$

where $\gamma < 0$. This implies that for all $x \in \mathbb{R}^d$, we get

$$\|\varphi(t, x)\| = \big\|e^{At}x\big\| \leq \big\|e^{At}\big\|\|x\| \leq Ke^{\gamma t}\|x\| \quad \text{for all } t \geq 0\,.$$

Hence, $x^* = 0$ is exponentially stable.

Exercise 30.

(i) Consider the function

$$\mu(t) := \int_{-\infty}^{t} e^{A(t-s)}g(s)\,\mathrm{d}s \quad \text{for all } t \in \mathbb{R}$$

from the hint. Firstly, we prove that the improper integral in the definition of μ exists. Note that the assumptions on the eigenvalues on A imply that

$$\big\|e^{At}\big\| \leq Ke^{\frac{a}{2}t} \quad \text{for all } t \geq 0\,,$$

due to Proposition 3.9, and there exists an $M > 0$ such that

$$\|g(t)\| \leq M \quad \text{for all } t \geq 0\,.$$

Hence for $t \in \mathbb{R}$ fixed and $\tau < t$, we get

$$\left\| \int_\tau^t e^{A(t-s)} g(s) \, \mathrm{d}s \right\| \leq \int_\tau^t \| e^{A(t-s)} \| \| g(s) \| \, \mathrm{d}s \leq \int_\tau^t KMe^{\frac{a}{2}(t-s)} \, \mathrm{d}s = -KM\frac{2}{a}\Big(1 - e^{\frac{a(t-\tau)}{2}}\Big) ,$$

and since this converges to $-KM\frac{2}{a}$ in the limit $\tau \to -\infty$, the integral used for the definition of the function μ exists. In addition, this calculation implies that the function μ is bounded, since $-KM\frac{2}{a}$ does not depend on $t \in \mathbb{R}$.

We show now that μ is a solution of the linear inhomogeneous differential equation. With $\mu(t) = e^{At} \int_{-\infty}^t e^{-As} g(s) \, \mathrm{d}s$, we get using the product rule that

$$\dot{\mu}(t) = Ae^{At} \int_{-\infty}^t e^{-As} g(s) \, \mathrm{d}s + e^{At} e^{-At} g(t) = A\mu(t) + g(t) ,$$

which shows that μ is indeed a solution of the inhomogeneous system.

(ii) Assume there is another bounded solution $\nu : \mathbb{R} \to \mathbb{R}^2$ of the inhomogeneous system. Then $\lambda : \mathbb{R} \to \mathbb{R}^2$, given by $\lambda(t) := \mu(t) - \nu(t)$ is a bounded solution of the homogeneous linear differential equation $\dot{x} = Ax$ (see proof of Proposition 3.10). But it is clear that the trivial solution is the only bounded solution of the homogeneous system (see also Exercise 24), so μ is the unique bounded solution of the inhomogeneous system.

(iii) Pick any two solutions $\gamma, \beta : \mathbb{R} \to \mathbb{R}^2$ of the inhomogeneous system. Then the difference $\alpha : \mathbb{R} \to \mathbb{R}^2$, given by $\alpha(t) := \gamma(t) - \beta(t)$, is a solution of the homogeneous system, and since the spectrum of the system $\Sigma(A)$ lies in the negative half line, we get using Proposition 3.9 that $\lim_{t\to\infty} \alpha(t) = 0 \in \mathbb{R}^2$, which implies that

$$\lim_{t\to\infty} \| \gamma(t) - \beta(t) \| = 0 .$$

This proves the claimed statement. Note that the function μ attracts all solutions forward in time, but so does every other solution of the system, and the function μ is not special in that sense. When looking at nonautonomous differential equation, there are essentially two ways to attract: *forward attraction* and *pullback attraction*. We have just observed that all solutions are forward attracting. Let us explore pullback attraction. A solution $\gamma : \mathbb{R} \to \mathbb{R}^2$ of the inhomogeneous system is pullback attractive if we have

$$\lim_{t_0\to-\infty} \| \lambda(t, t_0, x_0) - \gamma(t) \| = 0 \quad \text{for all } \ x_0 \in \mathbb{R}^2 ,$$

where λ denotes the general solution of the inhomogeneous system. Pullback attraction thus refers to attraction in the past, while forward attraction measures attraction in the future, and past and future are not related in general for nonautonomous differential equations. The interesting thing is that only one solution is pullback attractive, and this is μ. To see that μ is pullback attractive, we use the variation of constants formula, and we get

$$\lim_{t_0\to-\infty} \lambda(t, t_0, x_0) = \lim_{t_0\to-\infty} e^{A(t-t_0)} x_0 + \lim_{t_0\to-\infty} \int_{t_0}^t e^{A(t-s)} g(s) \, \mathrm{d}s = 0 + \mu(t) = \mu(t) \quad \text{for all } \ t \in \mathbb{R} .$$

Can you prove that μ is the only pullback attractive solution?