MATH50004 Differential Equations
Spring Term 2021/22
Problem Sheet 7

Exercise 31 (First steps in phase plane analysis).

Consider the two-dimensional differential equation

$$\dot{x} = -x\,,$$
$$\dot{y} = -2y + 2x^3\,.$$

(i) Show that $(0,0)$ is the only equilibrium.

(ii) Show that this equilibrium is attractive using linearisation.

(iii) Draw the nullclines in the plane (these are the curves where $\dot{x} = 0$ and $\dot{y} = 0$), and identify the following regions bounded by the nullclines and named using London postcode logic:
NE (where $\dot{x} > 0$, $\dot{y} > 0$), NW (where $\dot{x} < 0$, $\dot{y} > 0$), SW (where $\dot{x} < 0$, $\dot{y} < 0$), SE (where $\dot{x} > 0$, $\dot{y} < 0$).

(iv) Clarify for each of these four regions whether they are invariant, positively invariant or negatively invariant. Can this system have a periodic orbit? Try to answer the last question on an intuitive level rather than rigorously.

(v) Prove that the equilibrium $(0,0)$ is globally attractive, i.e. all points converge to it forward in time: $\lim_{t\to\infty} \varphi(t,x,y) = (0,0)$ for all $(x,y) \in \mathbb{R}^2$.

Hint. Analyse in (v), using also information obtained in (iv), how the flow can move between the different regions. For making your arguments rigorous, use Exercise 16 (ii).

Remark. Some of you may notice that there is a faster way to get to (v), but the objective of this exercise is to explore how to analyse two-dimensional nonlinear differential equations using tools from phase plane analysis by means of a relatively simple example first.

Exercise 32 (Stability under linear transformations).

Consider an autonomous differential equation

$$\dot{x} = f(x)\,,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous, and assume that x^* is a stable equilibrium. Let $T \in \mathbb{R}^{d\times d}$ be an invertible matrix. Show that then $y^* := T^{-1}x^*$ is a stable equilibrium of the differential equation

$$\dot{y} = g(y)\,,$$

where $g(y) := T^{-1}f(Ty)$.

Hint. Show first that $\lambda : I \to \mathbb{R}^d$ is a solution of $\dot{x} = f(x)$ if and only if $\mu : I \to \mathbb{R}^d$, given by $\mu(t) := T^{-1}\lambda(t)$, is a solution of $\dot{y} = g(y)$.

Exercise 33 (Openness of the domain of attraction).

Consider an autonomous differential equation

$$\dot{x} = f(x)\,,$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous and $D \subset \mathbb{R}^d$ is an open set. Show that the domain of attraction

$$W^s(x^*) := \left\{x \in D : \lim_{t\to\infty} \varphi(t,x) = x^*\right\}$$

of an attractive equilibrium $x^* \in D$ is an open subset of D.

Hint. You can use without proof that the flow is a continuous function.

Exercise 34 (Boundary of an invariant set).

Consider an autonomous differential equation

$$\dot{x} = f(x)\,,$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous on an open set $D \subset \mathbb{R}^d$, and assume that M is an invariant set. Show that then the boundary ∂M is also invariant.

Remark. We consider here the metric space D, as a subset of $\mathbb{R}^d$, inheriting the metric from $\mathbb{R}^d$. Here, the boundary ∂M of M needs to be taken in the topology of the open set D, meaning that we have $\partial M = \partial_{\mathbb{R}^d} M \cap D$, where $\partial_{\mathbb{R}^d} M$ is the boundary of M in $\mathbb{R}^d$.

Exercise 35 (Optional challenging question).

Consider $D := J \times \mathbb{R}^d$ with an open interval $J \subset \mathbb{R}$, and let the right hand side $f : D \to \mathbb{R}^d$ of the differential equation $\dot{x} = f(t,x)$ be continuous and locally Lipschitz continuous. Furthermore, assume that f is linearly bounded, i.e. there exist continuous functions $\rho, \sigma : J \to \mathbb{R}_0^+$ such that

$$\|f(t,x)\| \le \rho(t)\|x\| + \sigma(t) \quad \text{for all } \ (t,x) \in D\,.$$

Show that then the maximal solution to each initial value problem

$$\dot{x} = f(t,x)\,, \qquad x(t_0) = x_0\,,$$

with $(t_0, x_0) \in D$, exists globally, i.e. $I_{max}(t_0, x_0) = J$.

Hint. Use Proposition 2.1 and the Gronwall lemma (Lemma 4.9).

Comments on importance and difficulty of the exercises. Exercise 31 makes you familiar with techniques to analyse two-dimensional differential equations. The idea is to first understand where the flow is directed to and to study situations where this can be used to get a precise understanding of the asymptotic behaviour; earlier material developed in the exercises is important for this analysis, as pointed out. Analysing two-dimensional phase portraits is fun, and you will find lots of questions of this type in the past exams (and on the next problem sheet). This is not so easy when you do it the first time, but I have selected more elementary examples first. Exercise 32 shows that stability properties are preserved under linear transformations. We know this already when we transform a linear system in Jordan normal form, i.e. when the function f is linear. This exercise trains you deal with the ε and δ's of the stability definition. Exercises 33 and 34 are challenging, even though the solutions are not lengthy. The challenging Exercise 35 gives another application of the Gronwall lemma. This lemma was used before to prove the theorem on linearised stability (Theorem 4.10).