MATH50004 Differential Equations
Spring Term 2021/22
Problem Sheet 8

This is the last problem sheet, and since it covers material from both Week 10 and 11, it is a bit longer (eight exercises, and an optional challenging exercise). With the knowledge of Week 10, you can do the first four Exercises 36–39, and the optional challenging Exercise 44.

Exercise 36 (Omega and alpha limit sets).
Compute the omega and alpha limit sets of all points for the following differential equations:

$$\text{(i)}\ \dot{x} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} x\,, \quad \text{(ii)}\ \dot{x} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} x\,, \quad \text{(iii)}\ \dot{x} = \begin{pmatrix} 4 & 5 \\ -5 & 2 \end{pmatrix} x\,, \quad \text{(iv)}\ \dot{x} = x(x+1)\,.$$

Exercise 37 (Instability via scalar-valued functions with positive orbital derivative).
Consider the trivial equilibria of the two differential equations

$$\begin{aligned} \dot{x} &= -y + x^3\,, \\ \dot{y} &= x + y^3\,, \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= 2xy + x^3\,, \\ \dot{y} &= -x^2 + y^5\,. \end{aligned}$$

Can you make conclusions about their stability using linearisations? Use a scalar-valued function with positive orbital derivative for each differential equation in order to prove instability of the trivial equilibria.

Exercise 38 (Phase plane analysis I).
Consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= (1 - y - x)x\,, \\ \dot{y} &= (x - 2 - y)y\,, \end{aligned}$$

which is a population model with the prey population given by x and the predator population given by y. Although defined on $\mathbb{R}^2$, we restrict all analysis to the first quadrant, where $x, y \geq 0$.

(i) Draw the nullclines and the regions for the four directions, as in Exercise 31.

(ii) Determine the equilibria and their stability via linearisation.

(iii) Show that the predator population will die, i.e. given $(x, y) \in (0, \infty)^2$, the second component of $\varphi(t, x, y)$ converges to 0 as $t \to \infty$.

Exercise 39 (Phase plane analysis II).
Consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= -xy + x(1 - 2x)\,, \\ \dot{y} &= yx - y\,, \end{aligned}$$

which is defined on $\mathbb{R}^2$, but we restrict the analysis on the first quadrant, where $x, y \geq 0$. The flow of this differential equation will be denoted by φ.

(i) Determine the equilibria of this differential equation, draw the nullclines, and indicate the directions in the regions bounded by the nullclines.

(ii) Determine the type of each equilibrium: is it hyperbolic or non-hyperbolic? For hyperbolic equilibria, indicate whether it is attractive, repulsive or a saddle.

(iii) Prove that for any $R > 2$, the region

$$M_R := \{(x, y) \in [0, \infty)^2 : 0 \leq y \leq R - 1\,, x \geq 0 \text{ and } x + y \leq R\}$$

is positively invariant.

(iv) Show that $\sup J_{max}((x, y)) = \infty$ for all $(x, y) \in [0, \infty)^2$, i.e. all solutions in this quadrant exist globally forward in time.

(v) Show that for any $(x, y) \in (0, \infty)^2$, there exists a $t > 0$ such that $\varphi_1(t, x, y) \leq 1$. Here $\varphi_1(t, x, y)$ denotes the first component (the x-component) of the flow φ.

(vi) Show that this system does not have a periodic orbit in the first quadrant.

(vii) Show that $\lim_{t\to\infty} \varphi(t, x, y) = (\frac{1}{2}, 0)$ for all $(x, y) \in (0, \infty)^2$.

Exercise 40 (Lyapunov's direct method I).
Consider the trivial equilibrium $(0, 0)$ of the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= -y - x^3\,, \\ \dot{y} &= x - y^3\,. \end{aligned}$$

(i) Linearise the system in $(0, 0)$ and determine whether stability of this equilibrium can be concluded from this analysis.

(ii) Prove that $(0, 0)$ is asymptotically stable and determine the domain of attraction $W^s((0, 0))$.

Hint. Show that this differential equation has a Lyapunov function V. It is often useful to first consider an ansatz of the form $V(x, y) = ax^2 + bxy + cy^2$.

Exercise 41 (Lyapunov's direct method II).
Consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= -y^3\,, \\ \dot{y} &= x^3\,. \end{aligned}$$

Determine whether the equilibrium $(0, 0)$ is stable and/or asymptotically stable by showing that this differential equation admits a Lyapunov function. Draw the phase portrait.

Hint. Consider Lyapunov functions in form of polynomials of higher order.

Exercise 42 (Existence of a periodic orbit).
Consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -x - (2x^2 + 3y^2 - 1)y\,. \end{aligned}$$

(i) Show that the annulus $\left\{(x, y) \in \mathbb{R}^2 : \frac{1}{3} < x^2 + y^2 < \frac{1}{2}\right\}$ is positively invariant.

(ii) Show that there exists a periodic orbit.

Exercise 43 (Omega limit set consisting of equilibria and heteroclinic orbits).

Consider the two-dimensional differential equation.

$$\dot{x} = \sin(x)\big(-\tfrac{1}{10}\cos(x) - \cos(y)\big)\,,$$
$$\dot{y} = \sin(y)\big(\cos(x) - \tfrac{1}{10}\cos(y)\big)\,,$$

which is defined on $\mathbb{R}^2$, but we restrict the analysis on the square $[0,\pi]^2$.

(i) Determine the equilibria of this differential equation, linearise at each of the equilibria, and determine the local dynamical behaviour close to these equilibria.

(ii) Show that the omega limit set $\omega((x,y))$ for $(x,y) \in (0,\pi)^2 \setminus \{(\frac{\pi}{2},\frac{\pi}{2})\}$ is given by the boundary of the square $[0,\pi]^2$.

(iii) Draw the phase portrait of this differential equation in the square $[0,\pi]^2$.

Hint. For solving (ii), find a Lyapunov function that vanishes at the boundary of $[0,\pi]^2$ and is maximal at $(\frac{\pi}{2},\frac{\pi}{2})$.

Exercise 44 (Optional challenging question).

Consider an autonomous differential equation

$$\dot{x} = f(x)\,,$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous, defined on an open set $D \subset \mathbb{R}^d$. Suppose that for a given $x \in D$, the half-orbit $O^+(x)$ is bounded, and we have $\overline{O^+(x)} \subset D$. Show that $\omega(x)$ is connected.

Definition. We say that $M \subset \mathbb{R}^d$ is *disconnected* if there exist two disjoint nonempty and closed sets $M_1 \subset \mathbb{R}^d$ and $M_2 \subset \mathbb{R}^d$ such that

$$M_1 \cap M \neq \emptyset\,, \quad M_2 \cap M \neq \emptyset\,, \quad \text{and} \quad M \subset M_1 \cup M_2\,.$$

We call a set $M \subset \mathbb{R}^d$ *connected* if it is not disconnected.

Remark. This result is needed in the proof of the Poincaré–Bendixson theorem, see Extra Material 2.

Comments on importance and difficulty of the exercises. This last problem sheet is a little bit different in the sense that there are no theoretical questions. Exercise 36 is an elementary exercise to understand omega and alpha limit sets better. It also makes connections to what you have learned about linear systems and one-dimensional systems. Exercise 37 concerns an application of the orbital derivative in order to prove instability. A related analysis is carried out in Week 11, and Lyapunov's direct method gives stability or asymptotic stability, and this is trained in Exercises 41 and 42. Exercises 38 and 39 concern more advanced examples of how to analyse two-dimensional differential equations (in comparison to the introductory example studied in Exercise 31. Exercise 42 concerns a standard method to verify the existence of a periodic orbit, while Exercise 43 concerns an interesting example which shows that omega limit sets can be more complicated then just equilibria or periodic orbits. The Poincaré–Bendixson theorem shows that omega limit sets are of a very restrictive type in two dimensions, although there is much more variety than in dimension one. Here, the omega limit is given by four equilibria and connecting orbits. Note that not all variations of equilibria and connecting orbits can be omega limit sets. For instance, while it is not excluded in the formulation of the Poincaré–Bendixson theorem that a heteroclinic orbit, together with its two equilibria, is an omega limit set, this is not possible. You may want think of why this is the case as an additional exercise. The challenging Exercise 44 concerns connectivity of omega limit sets, which is needed in the proof of the Poincaré–Bendixson theorem.