Probability for Statistics
Revision Problem Sheet

1. Consider the probability space $(\Omega, \mathcal{F}, \Pr)$ with $A, B \in \mathcal{F}$. Using only the Kolmogorov axioms prove

   (a) $\Pr(A) \le 1$,

   (b) If $A \subseteq B$, then $\Pr(A) \le \Pr(B)$, and

   (c) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

2. Consider two coins, of which one is a normal fair coin and the other is biased so that the probability of obtaining a Head is $p > 1/2$.

   (a) Suppose $p = 1$ and a coin is selected at random and flipped n times, with flips mutually independent. Evaluate the conditional probability that the selected coin is the normal one, given that the first n flips are all Heads.

   (b) Now suppose $1/2 < p < 1$ and that again, one of the coins is selected randomly and flipped n times. Let E be the event that the n tosses result in k Heads and $n - k$ Tails, and let F be the event that the coin is fair. Find $\Pr(F|E)$.

3. Suppose X is a set containing n elements. If A and B are randomly chosen subsets of X, what is the probability that $A \subseteq B$?

4. Suppose a biased coin is flipped n times, independently. Let p denote the probability the coin lands with heads showing, and let the random variable X denote the number of heads shown.

   Using the probability generating function of X, determine expressions in terms of n and p for:

   (a) The mean and variance of X,

   (b) The probability that X is even,

   (c) The probability that X is divisible by 3.

Discussion questions

The next two questions are adapted from the work of Kahneman and Tversky, psychologists who studied subjective perceptions of probability. See the Additional Resources section on Blackboard!

5. A city contains two hospitals, one small and one large. The long-term proportion of boys born can be taken to be 50%. On a given day, the proportion of boys born in one of the hospitals is 55%. Which hospital is this more likely to be?

6. Tom is an opera buff who enjoys touring art museums when on holiday. Growing up, he enjoyed playing chess with family members and friends. Which of the following two situations is more likely?

   (a) Tom plays trumpet for a major symphony orchestra.

   (b) Tom is a farmer.

Table 1: Success rates for the treatment of kidney stones by open surgery (Treatment A) and percutaneous nephrolithotomy (Treatment B).

	Treatment A	Treatment B
Small stones	81 / 87 (93%)	234 / 270 (87%)
Large stones	192 / 263 (73%)	55 / 80 (69%)
Total	273 / 350 (78%)	289 / 350 (83%)

7. Consider the data in Table 1, comparing two treatments for kidney stones.

   Which treatment is better? Evaluate the following two responses:

   I. 83% of the time, Treatment B was successful, whereas Treatment A was successful only 78% of the time. So Treatment B is better.

   II. For patients with small stones, Treatment A was 93% successful whereas Treatment B was 87% successful. Similarly, for large stones, Treatment A was 73% successful, while treatment B was 69% successful. So Treatment A is better.