Probability for Statistics
Problem Sheet 1

The first three questions should be accessible once you have watched up to lecture 3. Questions 4 and 5 should be accessible once you have watched up to lecture 6. Question 6 is a skills question, extending ideas that should be familiar from last year. Question 7 is an optional discussion question, for interest.

1. Let Ω be a set.
   (a) Show that the collection $\mathcal{F} = \{\emptyset, \Omega\}$ is a sigma algebra.
   (b) Show that for any subset $E \subseteq \Omega$, $\mathcal{F}_E = \{\emptyset, E, E^c, \Omega\}$ is a sigma algebra.
   (c) Let $\mathcal{F}$ be the collection of all subsets of Ω. Show that $\mathcal{F}$ is a sigma algebra.
   (d) Show that the intersection of two sigma algebras on Ω is a sigma algebra.
   (e) Give an example to show that the union of two sigma algebras on Ω need not be a sigma algebra.

2. Suppose a fair coin is flipped repeatedly, and that flips are independent. Use the continuity property of the probability function $\Pr$ to show that, with probability 1, the coin will eventually land heads up.

3. Let $\Omega = [0, 1]$, the unit interval. Define $\mathcal{F}$ to be the collection of all countable or co-countable subsets of Ω, where a co-countable set is one whose complement is countable.
   (a) Show that $\mathcal{F}$ is a sigma algebra. *[Hint: Is a countable union of countable sets countable?]*
   (b) Define the function $P : \mathcal{F} \to [0, 1]$ by
   $$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is co-countable} \end{cases} .$$
   Determine whether or not P is countably additive.

4. Consider a probability space $(\Omega, \mathcal{F}, \Pr)$ in which
   $$\Omega = \{1, 2, 3, 4, 5, 6\}, \qquad \mathcal{F} = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}.$$
   Determine whether each of the two functions $X_1, X_2 : \Omega \to \mathbf{R}$ defined below is a random variable with respect to $\mathcal{F}$.
   $$X_1(s) = s, \qquad X_2(s) = \begin{cases} 0 & s \text{ even} \\ 1 & s \text{ odd} \end{cases} .$$

5. (a) Let $X : \Omega \to \mathbf{R}$ be a random variable, and let $\mathcal{B}$ be the Borel sigma algebra on $\mathbf{R}$. Show that $\mathcal{F}_X = \{X^{-1}(B) : B \in \mathcal{B}\}$ is a sigma algebra on Ω.
   (b) Consider an experiment in which a fair coin is flipped twice, so that the sample space is $\Omega = \{HH, HT, TH, TT\}$. Let $X : \Omega \to \mathbf{R}$ take the value 1 if precisely one flip comes up heads, and 0 otherwise. Determine the sigma algebra $\mathcal{F}_X$.

(c) For Ω as in the previous part, give an example of a function $Y : \Omega \to \mathbf{R}$ and a function g (with suitable domain) such that $X = g(Y)$ and $\mathcal{F}_X \subset \mathcal{F}_Y$.

6. (Review and extension of elementary probability.) In this question, you will derive the mean and variance of the hypergeometric distribution.

   (a) (Warm up) If $X \sim \text{BINOMIAL}(n, p)$, we can write $X = \sum_{i=1}^n Z_i$, where $Z_i \sim \text{BERNOULLI}(p)$ are independent. Use this representation to show that $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$.
   Suppose now that X is hypergeometric, representing the distribution of the number of red balls in a sample of size n drawn without replacement from an urn containing r red and w white balls, $N = r + w$. In this case,

$$\Pr(X = x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}.$$

   As in the binomial case, we can represent X as a sum of Bernoulli variables: $X = \sum_{i=1}^n Z_i$, where Z_i takes the value 1 if the ith ball is red and 0 otherwise.

   (b) What is the distribution of the Z_i ? Are they independent?

   (c) Show that $\mathrm{E}(X) = n\frac{r}{N}$.

   (d) (Harder) Show that $\mathrm{Var}(X) = n\frac{r}{N}\frac{w}{N}\frac{N-n}{N-1}$.

Optional question for group discussion

7. For real numbers $s > 1$, define the Riemann zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

   Let $s > 1$ be fixed, and let the random variable X have probability mass function

$$f_X(x) = \Pr(X = x) = \frac{1}{x^s}\frac{1}{\zeta(s)}, \qquad x \geq 1.$$

   Let D_k by the event that X is divisible by k, for $k \geq 2$.

   (a) What is $\Pr(D_k)$?

   (b) Show that the events $\{D_p : p \text{ is prime}\}$ are independent.

   (c) Prove Euler's formula for the zeta function in terms of the prime numbers:

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

   Hint: You may assume that whenever a collection $\{A_i : i \in I\}$ of events is independent, so is the collection $\{A_i^c : i \in I\}$. Recall also that for a countable collection of independent events,

$$\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = \prod_{i=1}^{\infty} \Pr(A_i).$$