MATH50010: Probability for Statistics
Problem Sheet 4

1. The joint pdf of the random variables X_1 and X_2 is
$$f_{X_1,X_2}(x_1,x_2) = k\exp\left\{-\left(\frac{x_1^2}{6} - \frac{x_1x_2}{3} + \frac{2x_2^2}{3}\right)\right\}, \text{ for } -\infty < x_1, x_2 < \infty.$$
Find $\mathrm{E}(X_1), \mathrm{E}(X_2), \mathrm{Var}(X_1), \mathrm{Var}(X_2), \mathrm{Cov}(X_1,X_2)$ and k.

2. Suppose
$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{N}_2\left[\mu = \begin{pmatrix} 2 \\ -5 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 4 \end{pmatrix}\right].$$
Compute $\Pr(X_1 > 0)$ and $\Pr(X_2 < -6)$.

3. Suppose X_1, X_2, and X_3 are iid $\mathrm{N}(1,1)$ random variables. Let $X_4 = 2X_2 + 2X_3$ and $X_5 = X_2 - 2X_3$.
   (a) Find the joint pdf of (X_1, X_4, X_5).
   (b) Find the marginal pdf of X_5.

4. Suppose X and Y are two random variables each with finite mean and variance. Prove $-1 \leq \rho_{XY} \leq 1$ by using the fact that
$$\mathrm{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) \text{ and } \mathrm{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right)$$
are both positive quantities.

5. Suppose that U_1 and U_2 are independent and identically distributed $\mathrm{Unif}(0,1)$ random variables. Let random variables Z_1 and Z_2 be defined by
$$Z_1 = \sqrt{-2\log(U_1)}\cos(2\pi U_2),$$
$$Z_2 = \sqrt{-2\log(U_1)}\sin(2\pi U_2).$$
Find the joint pdf of (Z_1, Z_2).

6. Suppose $(X_1, \ldots, X_n)$ is a collection of independent and identically distributed random variables taking values on $\mathbb{X}$ with pmf/pdf f_X and cdf F_X. Let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from $(X_1, \ldots, X_n)$, that is
$$Y_n = \max\{X_1, \ldots, X_n\}, \quad Z_n = \min\{X_1, \ldots, X_n\}.$$
   (a) Show that the cdfs of Y_n and Z_n are given by
$$F_{Y_n}(y) = \{F_X(y)\}^n, \quad F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n.$$
   (b) Suppose $X_1, \ldots, X_n \sim \mathrm{Unif}(0,1)$, that is
$$F_X(x) = x, \text{ for } 0 \leq x \leq 1.$$
   Find the cdfs of Y_n and Z_n.

(c) Suppose $X_1, \ldots, X_n$ have cdf

$$F_X(x) = 1 - x^{-1}, \text{ for } x \geq 1.$$

Find the cdfs of Z_n and $U_n = Z_n^n$.

(d) Suppose $X_1, \ldots, X_n$ have cdf

$$F_X(x) = \frac{1}{1 + e^{-x}}, \text{ for } x \in \mathbb{R}.$$

Find the cdfs of Y_n and $U_n = Y_n - \log n$.

(e) Suppose $X_1, \ldots, X_n$ have cdf

$$F_X(x) = 1 - \frac{1}{1 + \lambda x}, \text{ for } x > 0.$$

Find the cdfs of Y_n, Z_n, $U_n = Y_n/n$, and $V_n = nZ_n$.

For discussion

7. Let $X_1, \dots X_n \sim \textsc{Uniform}(0,1)$ and let $M_n = \max\{X_1, \dots X_n\}$.

   (a) Show that for $\epsilon > 0$,

   $$\Pr(M_n < 1 - \epsilon) = (1 - \epsilon)^n.$$

   (b) Use the result above to show that for all $\epsilon > 0$

   $$\lim_{n \to \infty} \Pr(|M_n - 1| \geq \epsilon) = 0.$$

   Later we will say that this shows that the random variable M_n converges in probability to the constant value 1.

   (c) Now (by taking $\epsilon = \frac{t}{n}$), show that the distribution function of the rescaled variable $n(1-M_n)$ converges to the CDF of a known distribution.

8. Suppose Y and $\boldsymbol{X} = (X_1, X_2)^\top$ jointly follow a trivariate normal distribution. Here Y is a univariate random variable and $\boldsymbol{Z} = (Y, X_1, X_2)^\top$ is a (3×1) trivariate normal random vector with mean

   $$\boldsymbol{\mu} = \begin{pmatrix} \mu_Y \\ \boldsymbol{\mu_X} \end{pmatrix} \text{ and variance-covariance matrix } \boldsymbol{M}^{-1} = \begin{pmatrix} m_{YY} & \boldsymbol{M}_{Y\boldsymbol{X}} \\ \boldsymbol{M}_{Y\boldsymbol{X}}^\top & \boldsymbol{M}_{\boldsymbol{XX}} \end{pmatrix}^{-1},$$

   where μ_Y is the univariate mean of Y, $\boldsymbol{\mu_X}$ is the (2×1) mean vector of $\boldsymbol{X}$, $\boldsymbol{\mu}$ is the (3×1) mean vector of both $\boldsymbol{X}$ and Y, m_{YY} is the first diagonal element of $\boldsymbol{M}$, $\boldsymbol{M}_{\boldsymbol{XX}}$ is the lower-right (2×2) submatrix of $\boldsymbol{M}$, and $\boldsymbol{M}_{Y\boldsymbol{X}}$ is the remaining off-diagonal (1×2) submartix of $\boldsymbol{M}$. (Note that we parameterize the multivariate normal in terms of the *inverse* of its variance-covariance matrix. This will significantly simplify calculations!)

   (a) Derive the conditional distribution of Y given both X_1 and X_2. *[Hint: Use vector/matrix notation.]*

   (b) Now suppose Y and $\boldsymbol{X} = (X_1, \dots, X_n)^\top$ jointly follow a multivariate normal distribution. Here Y remains a univariate random variable and $\boldsymbol{Z} = (Y, X_1, \dots, X_n)^\top$ is an $[(n+1) \times 1]$ multivariate normal random vector. Use the same notation for the mean and the inverse of the variance-covariance matrix, but with appropriately adjusted dimensions. Derive the conditional distribution of Y given $X_1, \dots, X_n$. *[Hint: If you used vector/matrix notation in part (a), this problem will be very easy. If you did not, it will be very hard!]*

   (c) Set $n = 1$ and check that your answer is the same as the conditional distribution for the bivariate normal derived in lecture.