Probability for Statistics MATH50010
Unseen Problem 1

Let Ω be a finite set. Does there exist an event space, i.e. an algebra of sets $\mathcal{F}$ on Ω, with precisely six elements?

More generally, for what positive integer values N does there exist an algebra of sets of size N?

Solution

We will show the following general result: if $|\Omega| = n \geq 1$*, then the sigma algebras on* Ω *have size* 2^k *for some* $1 \leq k \leq n$*.*

Let $\mathcal{F}$ *be an algebra on* Ω*. For each* $x \in \Omega$*, define*

$$A_x = \bigcap_{B \in \mathcal{F}: x \in B} B,$$

the intersection of all sets in $\mathcal{F}$ *containing* x*. By construction, each* $A_x \in \mathcal{F}$*.*

Key point: *Think back to the idea of an event space as representing the observable outcomes of an experiment. The experiment may not be able to distinguish between all outcomes: e.g. considering the 12 balls problem, the experiment that consists of weighing* $\{1, 2, 3, 4\}$ *against* $\{5, 6, 7, 8\}$ *cannot distinguish between outcomes* 1 *and* 2*. So, for this experiment, for any event* $E \in \mathcal{F}$*, either both* $1, 2 \in E$ *or* $1, 2 \notin E$*. The event* A_x *defined above is the collection of all outcomes that cannot be distingished from* x*. Below, we will show that sets of the form* A_x *partition* Ω*.*

Note that for $x, y \in \Omega$*, either* $A_x = A_y$ *or* $A_x \cap A_y = \emptyset$*. To see this, suppose for contradiction that there exists* $z \in A_x \cap A_y$ *but* $y \notin A_x$*. Then for some* $B \in \mathcal{F}$*, we have that* $x \in B$ *but* $y \notin B$*. But then* $y \in B^c$ *so that* $A_y \subseteq B^c$ *(since* $B^c \in \mathcal{F}$*), giving*

$$z \in A_x \cap A_y \subseteq A_y \subseteq B^c.$$

But $z \in A_x$ *so* $z \in B$*, a contradiction. Hence* $y \in A_x$ *so that* $A_y \subseteq A_x$*, and so by symmetry in fact* $A_x = A_y$*.*

It follows that the set Ω *is partitioned by sets* $A_{x_1}, \ldots A_{x_k}$ *for distinct elements* $x_1, \ldots x_k \in \Omega$*. This partition corresponds to the equivalence relation* $\sim$ *on* Ω *where* $x \sim y$ *if and only if there is no* $E \in \mathcal{F}$ *that distinguishes between* x *and* y*, i.e. no* $E \in \mathcal{F}$ *has* $x \in E$ *and* $y \notin E$ *or* $y \in E$ *and* $x \notin E$*.*

Now for any non-empty $B \in \mathcal{F}$*, if* $x \in B$*, we must in fact have* $A_x \subseteq B$*, so that each non-empty* B *can be written as*

$$B = A_{y_1} \cup \ldots \cup A_{y_l}$$

for distinct elements $y_1, \ldots y_l \in \{x_1, \ldots x_k\}$*. Hence there are* 2^k *distinct elements of* $\mathcal{F}$*, corresponding to the subsets of* $\{x_1, \ldots x_k\}$*.*

In conclusion, only N *of the form* 2^k *for some positive integer* k *can be the size of an algebra of sets. Moreover, for a set* Ω *with* $|\Omega| = n \geq 1$*, the above argument shows how to construct an algebra of sets of size* 2^k *for any* $1 \leq k \leq n$*: simply partition* Ω *into* k *equivalence classes.*