Probability for Statistics
Unseen Problem 2

Suppose X and Y are random variables on a probability space $(\Omega, \mathcal{F}, \Pr)$. Verify that the following are random variables. *You may find it easier to verify the necessary and sufficient condition given in Proposition 2.9 for a function to be a random variable.*

1. $T = X + c$ for c constant.
2. $U = X^2$.
3. $V = \min(X, Y)$.
4. (harder) $W = X + Y$. *Hint: if $X + Y > z$ then $X > z - Y$. Between two distinct real numbers there exists a rational number.*
5. $Z = XY$.

X is a random varible so for all $x \in \mathbf{R}$,

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}.$$

Now,

1. *For any $t \in \mathbf{R}$,*

$$\{\omega \in \Omega : T(\omega) \leq t\} = \{\omega \in \Omega : X(\omega) \leq t - c\} \in \mathcal{F}.$$

2. *For any $u \in \mathbf{R}$,*

$$\begin{aligned}\{\omega \in \Omega : U(\omega) \leq u\} &= \{\omega \in \Omega : X(\omega) \in \left[-\sqrt{u}, \sqrt{u}\right]\} \\ &= \{\omega \in \Omega : X(\omega) < -\sqrt{u}\}^c \cap \{\omega \in \Omega : X(\omega) \leq \sqrt{u}\} \in \mathcal{F}.\end{aligned}$$

3. *Note that $\min\{X, Y\} \leq v$ if and only if either $X \leq v$ or $V \leq v$. Hence we can write the event $\{\omega \in \Omega : V(\omega) \leq v\}$ as a union of two events*

$$\{\omega \in \Omega : V(\omega) \leq v\} = \{\omega \in \Omega : X(\omega) \leq v\} \cup \{\omega \in \Omega : Y(\omega) \leq v\} \in \mathcal{F}.$$

Hence V is a random variable.

4. *Note that $\{X + Y \leq z\} = \{X + Y > z\}^c$. Then $X + Y > z$ if and only if $X > z - Y$, which is true if and only if there exists $q \in \mathbf{Q}$ such that $X > q$ and $q > z - Y$. So then*

$$\{X + Y > z\} = \bigcup_{q \in \mathbf{Q}} \{X > q\} \cap \{Y > z - q\}.$$

Since X and Y are random variables, $\{X > q\} = \{\omega \in \Omega : X(\omega) > q\} \in \mathcal{F}$ and similarly $\{Y > z - q\} \in \mathcal{F}$, so that $\{X + Y > z\}$ is a countable union of sets in $\mathcal{F}$, and hence also in $\mathcal{F}$.

5. *$XY = \frac{1}{2}\left((X + Y)^2 - X^2 - Y^2\right)$, so this follows by combining results of earlier parts.*