MATH50010 - Probability for Statistics
Unseen Problem 5

Consider a rod of unit length. The rod is broken at two points, whose locations can be modelled as independent, uniformly distributed random variables.

1. What is the density function of the *ordered* breakpoints $(x_{(1)}, x_{(2)})$, where $x_{(1)} < x_{(2)}$?
2. What is the probability that the three segments of the rod fit together to form a triangle?

1.

$$f(x_{(1)}, x_{(2)}) \begin{cases} 2 & 0 < x_{(1)} < x_{(2)} < 1, \\ 0 & \textit{otherwise} \end{cases}$$

2. The three segments form a triangle precisely when no segment is larger than the sum of the other two (think of the triangle inequality). Equivalently, if no segment has length $\geq \frac{1}{2}$*. this is the same as the event*

$$\{X_{(1)} < \frac{1}{2}\} \cap \{X_{(2)} - X_{(1)} < \frac{1}{2}\} \cap \{1 - X_{(2)} < \frac{1}{2}\}.$$

Compute this from the joint density above, by conditioning of the value of $x_{(1)}$ *:*

$$\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{x_{(1)}+\frac{1}{2}} 2\, dx_{(2)} dx_{(1)} = \int_0^{\frac{1}{2}} 2x_{(1)}\, dx_{(1)} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Nice geometric interpretations of this well-known result are possible. Can you find one?