

MATH50010 - Probability for Statistics

Unseen Problem 7

Suppose a flea hops randomly on the vertices of a triangle, with all jumps equally likely. Find the probability that after n hops, the flea is back where it started.

Let $p_{11}^{(n)} = \Pr(X_n = 1 | X_0 = 1)$ be the probability of interest. Then, conditioning on X_1 ,

$$\begin{aligned} p_{11}^{(n)} &= \frac{1}{2} \Pr(X_n = 1 | X_1 = 2, X_0 = 1) + \frac{1}{2} \Pr(X_n = 1 | X_1 = 3, X_0 = 1) \\ &= \frac{1}{2} \Pr(X_n = 1 | X_1 = 2) + \frac{1}{2} \Pr(X_n = 1 | X_1 = 3) = \frac{1}{2} p_{21}^{(n-1)} + \frac{1}{2} p_{31}^{(n-1)} \end{aligned}$$

Now by symmetry, $p_{21}^{(n-1)} = p_{31}^{(n-1)}$, so that the above simplifies to

$$p_{11}^{(n)} = p_{21}^{(n-1)}.$$

Moreover, by the law of total probability

$$p_{21}^{(n-1)} + p_{22}^{(n-1)} + p_{23}^{(n-1)} = 1$$

and again by symmetry this gives

$$2p_{21}^{(n-1)} + p_{22}^{(n-1)} = 1.$$

This then gives

$$p_{11}^{(n)} = \frac{1}{2}(1 - p_{22}^{(n-1)}) = \frac{1}{2}(1 - p_{11}^{(n-1)}),$$

since symmetry gives $p_{11}^{(n-1)} = p_{22}^{(n-1)}$. We can clearly see that $p_{11}^{(n)} \rightarrow \frac{1}{3}$ as $n \rightarrow \infty$ because the states are symmetrical. Another way to see this is to set $p_{11}^{(n)} = p_{11}^{(n-1)} = p$ and solve the resulting equation $p = \frac{1}{2}(1 - p)$ to get $p = \frac{1}{3}$.

Then note that

$$\frac{1}{3} - p_{11}^{(n)} = \frac{1}{3} - \frac{1}{2}(1 - p_{11}^{(n-1)}) = \frac{1}{2} \left(p_{11}^{(n-1)} - \frac{1}{3} \right).$$

Iterating this argument gives

$$\frac{1}{3} - p_{11}^{(n)} = \left(\frac{-1}{2} \right)^n \left(1 - \frac{1}{3} \right).$$

This then gives

$$p_{11}^{(n)} = \frac{1}{3} + \frac{2}{3} \left(\frac{-1}{2} \right)^n.$$

Check: $p_{11}^0 = 1$ and $p_{11}^1 = 0$ as it should be.

Alternatively, we seek the diagonal entries of powers of the transition matrix.

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

This is not too bad if we use symmetry: we know that all the diagonal entries of P^n must be equal: clearly $p_{11}^{(n)} = p_{22}^{(n)} = p_{33}^{(n)}$. This means all we need to do is take the average of the trace of P^n (recall the trace is the sum of the diagonal entries).

Now the trace of P^n is also the sum of its eigenvalues. And the eigenvalues of P^n are all of the form λ^n where λ ranges over the eigenvalues of P .

So we compute the eigenvalues of P . For ease of notation we'll work with $Q = 2P$

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$\det(Q - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = \det \begin{pmatrix} 0 & 1 + \lambda & 1 - \lambda^2 \\ 0 & -1 - \lambda & 1 + \lambda \\ 1 & 1 & -\lambda \end{pmatrix}$$

Evaluating this determinant then gives the characteristic polynomial

$$(1 + \lambda)^2 + (1 - \lambda^2)(1 + \lambda) = (1 + \lambda)(2 + \lambda - \lambda^2) = (1 + \lambda)(1 + \lambda)(2 - \lambda),$$

giving the eigenvalues of Q as 2 and -1 (twice). The eigenvalues of P are then 1 and $-\frac{1}{2}$ (twice). The probability we seek is then the average of the trace of P^n , which is exactly

$$\frac{1}{3} \left(1 + 2 \left(\frac{-1}{2} \right)^n \right).$$

Happily this agrees with the first method.