Imperial College London

Problem Sheet 8

MATH50011
Statistical Modelling 1

Week 9

Lecture 15: Multivariate Normal Distributions

1. Let X and B be independent random variables such that $X \sim N(0,1)$ and $B \in \{-1,1\}$ with $P(B=1) = P(B=-1) = \frac{1}{2}$. Let $Z = XB$.
   (a) Find $Cov(X,Z)$.
   (b) Show that $Z \sim N(0,1)$.
   (c) Are X and Z independent?

2. Suppose $X \sim N\left(\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)$.
   (a) What is the distribution of $Z = \begin{pmatrix}1 & 1\\0 & 1\\1 & 0\end{pmatrix} X + \begin{pmatrix}-1\\-3\\2\end{pmatrix}$?
   (b) Are any of the components of Z independent?
   (c) Let $Y \sim N\left(\begin{pmatrix}2\\3\\2\end{pmatrix}, \begin{pmatrix}2 & 1 & 0\\1 & 2 & 0\\0 & 0 & 9\end{pmatrix}\right)$. What components of Y are independent?

3. Let
$$\begin{pmatrix}Y\\X\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_Y\\\mu_X\end{pmatrix}, \begin{pmatrix}\sigma_Y^2 & \rho\sigma_Y\sigma_X\\\rho\sigma_Y\sigma_X & \sigma_X^2\end{pmatrix}\right).$$
   (a) Find the conditional distribution of $Y|X=x$ (it will be a univariate normal distribution).

(b) Express the conditional mean $E(Y|X = x)$ as a linear function $\beta_0+\beta_1 x$. What are β_0 and β_1 in terms of the parameters of the bivariate normal distribution?

Lecture 16: Distributions and Independence Results

4. In the lecture we had the following definition:

   Let $Z \sim N(\mu, I_n)$, where $\mu \in \mathbb{R}^n$. $U = Z^T Z$ is said to have a *non-central χ^2-distribution* with n degrees of freedom (d.f.) and non-centrality parameter $\delta = \sqrt{\mu^T \mu}$. Notation: $U \sim \chi_n^2(\delta)$.

   (a) Show that the $\chi_n^2(\delta)$-distribution depends on μ only through δ.

   (b) Show that $E(U) = n + \delta^2$ and $Var(U) = 2n + 4\delta^2$.

   (c) Show that if $U_i \sim \chi_{n_i}^2(\delta_i)$, $i = 1, \ldots, k$, and $U_1, \ldots, U_k$ are independent then $\sum_{i=1}^k U_i \sim \chi^2_{\sum n_i}(\sqrt{\sum \delta_i^2})$.

   Hint: Use moment-generating functions.

5. In the lectures, we showed that for a sequence $T_n \sim t_n(0)$, $T \to_d N(0, 1)$. Similar results can be derived for the χ_n^2 and $F_{m,n}$ distributions.

   (a) Let $Z_1, \ldots, Z_n$ be iid $N(0, 1)$ and define $U_n = \sum_i Z_i^2$. Use large sample properties of U_n to derive a normal approximation to the χ_n^2 distribution.

   (b) For m fixed and $n \to \infty$, show that $F_n \sim F_{m,n}$ converges in distribution to a χ_m^2 random variable.

6. Revise the proofs of Lemmas 15-19 and the Fisher-Cochran theorem.