Applied Probability Concise Notes

MATH60045/70045

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Content from prior years assumed to be known.

Mathematics
Imperial College London
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January 10, 2023

Contents

3 Discrete-time Markov Chains

3.1 Definition of discrete time Markov Chains

Definition 3.1.1. *A discrete-time stochastic process $X = \{X_n\}_{n \in \mathbb{N}_0}$ taking values in countable state space E a Markov chain if it satisfies the Markov condition*

$$P(X_n = j \mid X_{n-1} = i, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0) = P(X_n = j \mid X_{n-1} = i), \forall n \in \mathbb{N}\ \forall x_0, \ldots, x_{n-2}, i, j \in E$$

Definition 3.1.2. *(Time Homogenous)*

1. *Markov Chain $\{X_n\}_{n \in \mathbb{N}_0}$ is time-homogenous if*

$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i),\ \forall n \in \mathbb{N}_0, i, j \in E$$

2. *Transition matrix $P = (p_{ij})_{i,j \in E}$ is the $K \times K$ matrix of transition probabilities*

Definition 3.1.3. *(Stochastic Matrix)*
A square matrix P a stochastic matrix if

1. $p_{ij} \geq 0, \forall i, j$
2. $\sum_j p_{ij} = 1\ \forall i$

Theorem 3.1.4. *Transition matrix P is stochastic*

3.2 The n-step transition probabilities and Chapman-Kolmogorov equations

Definition 3.2.1. *$n \in \mathbb{N}$, we have*

$$P_n = (p_{ij}(n)) = P(X_{m+n} = j, X_m = i),\ m \in \mathbb{N}_0$$

The matrix of n-step transition probabilities.

Lemma 3.2.2. *For discrete markov chain $\{X_n\}_{n \geq 0}$ on state space E we have*

$$P(X_{n+m} = x_{n+m} | X_n = x_n, \ldots, X_0 = x_0) = P(X_{n+m} = x_{n+m} \mid X_n = x_n),\ m \in \mathbb{N}, \forall x_{n+m}, x_n, \ldots, x_0 \in E$$

Theorem 3.2.3. *Let $m \in \mathbb{N}_0, n \in \mathbb{N}$ Then we have $\forall i, j \in E$*

$$p_{ij}(m+n) = \sum_{l \in E} p_{il}(m) p_{lj}(n) \quad P_{m+n} = P_m P_n \quad P_n = P^n$$

Remark 3.2.4. *Extend definition for case $K = \infty$*
Let $\mathbf{x}$ a K-dimensional row vector, P a $K \times K$ matrix

$$(\mathbf{x}P)_j := \sum_{i \in E} x_i p_{ij}, \quad (P^2)_{ik} := \sum_{j \in E} p_{ij} p_{jk},\ i, j, k \in \mathbb{N}$$

Define P^n similarly and take $(P^0)_{ij} = \delta_{ij}$

3.3 Dynamics of a Markov Chain

Definition 3.3.1. *Denote probability mass function of X_n for $n \in \mathbb{N}_0$ by*

$$\nu_i^{(n)} = P(X_n = i),\ i \in E$$

Take $K = card(E)$, denote by $\nu^{(n)}$ the K-dimensional row vector with elements $\nu_i^n, i \in E$
Call this the ***marginal distribution*** *of chain at time $n \in \mathbb{N}_0$*

Theorem 3.3.3. *We have*

$$\nu^{(m+n)} = \nu^{(m)} P_n = \nu^{(m)} P^n,\ \forall n \in \mathbb{N}, m \in \mathbb{N}_0$$

So

$$\nu^{(n)} = \nu^{(0)} P_n = \nu^{(0)} P^n,\ \forall n \in \mathbb{N}$$

Theorem 3.3.4. *Let $X = \{X_n\}_{n\in\mathbb{N}_0}$ a Markov chain on countable state space E*
Then given initial distribution $\nu^{(0)}$ and transition matrix P, we determine all finite dimensional distributions of Markov chain.
$\forall 0 \le n_1 < n_2 < \cdots < n_{k-1} < n_k$ $(n_i \in \mathbb{N}_0, i = 1, \dots, k), k \in \mathbb{N}, x_1, \dots, x_k \in E$ *We have*

$$\begin{aligned} P(X_{n_1} = x_1, X_{n_2} = x_2, \dots, X_{n_k} = x_k) &= (\nu^{(0)} P^{n_1})_{x_1} (P^{n_2-n_1})_{x_1x_2} \cdots (P^{n_k - n_{k-1}}) x_{k-1} x_k \\ &= (\nu P^{n_1}) x_1 p_{x_1x_2}(n_2 - n_1) \cdots p_{x_{k-1}x_k}(n_k - n_{k-1}) \end{aligned}$$

3.4 First passage/hitting times

Definition 3.4.1. *Define* ***first passage/hitting time*** *of X for state $j \in E$ as*

$$T_j = \min\{n \in N : X_n = j\}$$

If $X_n \neq j, \forall n \in \mathbb{N}$ then set $T_j = \infty$

Definition 3.4.2. *For $i, j \in E, n \in \mathbb{N}$ define* ***first passage probability***

$$f_{ij}(n) = P(T_j = n \mid X_0 = i) = P(X_n = j, X_{n-1} \neq j, \dots, X_1 \neq j \mid X_0 = i)$$

Probability that we visit state j at time n, given we start at i at time 0
Define $f_{ij}(0) = 0, f_{ij}(1) = p_{ij}, \forall i, j \in E$

Definition 3.4.4. *Define*

$$f_{ij} = P(T_j < \infty \mid X_0 = i)$$

For $i \neq j$, we have f_{ij} the probability that the chain ever visits state j, starting at i
Call f_{ii} the ***returning probability***

Proposition 3.4.5. $\forall i, j \in E$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

Lemma 3.4.7. $\forall i, j \in E, n \in \mathbb{N}$*, we have*

$$\begin{aligned} p_{ij}(n) &= \sum_{l=0}^{n} f_{ij}(l) p_{jj}(n-l) \\ &= \sum_{l=1}^{n} f_{ij}(l) p_{jj}(n-l) \end{aligned}$$

3.5 Recurrence and transience

Definition 3.5.1. *Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a markov chain on countable state space E.*

$$j \in E,\ P(X_n = j, \text{for some } n \in \mathbb{N} \mid X_0 = j) = f_{jj} \begin{cases} 1, & \text{recurrent;} \\ < 1, & \text{transient .} \end{cases}$$

Theorem 3.5.2. *$j \in E$*

$$\sum_{n=1}^{\infty} p_{ij}(n) = \begin{cases} \infty, & \iff \text{recurrent ;} \\ < \infty, & \iff \text{transient .} \end{cases}$$

Define

$$N_j = \sum_{n=0}^{\infty} I_n^{(j)}, \quad I_n^{(j)} = I_{X_n=j} = \begin{cases} 1, & \text{if } X_n = j; \\ 0, & \text{if } X_n \neq j. \end{cases}$$

Theorem 3.5.3. *$j \in E$ transient*

1. $P(N_j = n \mid X_0 = j) = f_{jj}^{n-1}(1 - f_{jj})$ *for $n \in \mathbb{N}$ geometric distribution with param f_{jj}*
2. $i \neq j$

$$P(N_j = n \mid X_0 = i) = \begin{cases} 1 - f_{ij}, & \text{if } n = 0; \\ f_{ij} f_{jj}^{n-1}(1 - f_{jj}), & \text{if } n \in \mathbb{N}. \end{cases}$$

Corollary 3.5.4. *$j \in E$ transient*

1.

$$E(N_j \mid X_0 = j) = \frac{1}{1 - f_{jj}}$$

2. *$i \neq j$ we have*

$$E(N_j \mid X_0 = i) = \frac{f_{ij}}{1 - f_{jj}}$$

Theorem 3.5.5. *Given $X_0 = j$, we have*

$$E(N_j \mid X_0 = j) = \sum_{n=0}^{\infty} p_{jj}(n)$$

Sum may diverge to ∞

Corollary 3.5.6. *$j \in E$ transient then $p_{ij}(n) \xrightarrow[n\to\infty]{} 0, \forall i \in E$*

3.5.1 Mean recurrence time, null and positive recurrence

Definition 3.5.7. *The **mean recurrence time** μ_i of state $i \in E$ defined as $\mu_i = E[T_i \mid X_0 = i]$*

Theorem 3.5.8. *Let $i \in E$. We have $P(T_i = \infty \mid X_0 = i) > 0 \iff i$ transient, where we get*

$$\mu_i = E[T_i \mid X_0 = i = \infty]$$

Theorem 3.5.9. *For recurrent state $i \in E$ we have*

$$\mu_i = E[T_i \mid X_0 = i] = \sum_{n=1}^{\infty} n f_{ii}(n)$$

Can be finite or infinite.

Definition 3.5.10. *A recurrent state $i \in E$*

$$\mu_i = \begin{cases} \infty, & \text{called } \textbf{null}; \\ < \infty, & \text{called } \textbf{positive}. \end{cases}$$

Theorem 3.5.11. *Recurrent state $i \in E$ null $\iff p_{ii}(n) \xrightarrow[n\to\infty]{} 0$*
Further, if this holds, then $p_{ji}(n) \xrightarrow[n\to\infty]{} 0, \forall j \in E$

3.5.2 Generating functions for $p_{ij}(n), f_{ij}(n)$ (READING MATERIAL)

3.5.3 Example: Null recurrence/transience of a simple random walk (READING MATERIAL)

SEE FULL OFFICIAL NOTES

3.6 Aperiodicity and ergodicity

Definition 3.6.1. *Period of state i defined by*

$$d(i) = gcd\{n : p_{ii}(n) > 0\}$$

Definition 3.6.4. *A state ergodic if it is positive recurrent and aperiodic*

3.7 Communicating classes

Definition 3.7.1. *(Accessible and Communicating)*

1. *j accessible from i, $i \to j$, if $\exists m \in \mathbb{N}_0$ s.t $p_{ij}(m) > 0$*
2. *i, j communicate, if $i \to j$ and $j \to i$; write $i \leftrightarrow j$*

Theorem 3.7.2. *(Communication an equivalence relation)*
Satisfies, reflexivity, symmetry and transitivity

Theorem 3.7.4. *If $i \leftrightarrow j$ then*

1. *i, j have same period*
2. *i transient/recurrent $\iff$ j transient/recurrent*
3. *i null recurrent $\iff$ j null recurrent*

Definition 3.7.5. *Set of states C is*

1. ***closed** if $\forall i \in C, j \notin C, p_{ij} - 0$*
2. ***irreducible** if $i \leftrightarrow j, \forall i, j \in C$*

Theorem 3.7.6. *Let C a closed communicating class, transition matrix P restricted to C is stochastic*

3.7.1 The decomposition theorem

Theorem 3.7.8. *C a communicating class, consisting of recurrent states. Then C is closed*

Theorem 3.7.9. *State-space E can be partitioned uniquely into*

$$E = \underbrace{T}_{\text{transient states}} \cup \left(\bigcup_i \underbrace{C_i}_{\substack{\text{irreducible, closed} \\ \text{set of recurrent states}}} \right)$$

Theorem 3.7.11. *$K < \infty$ Then at least one state is recurrent and all recurrent states are positive.*

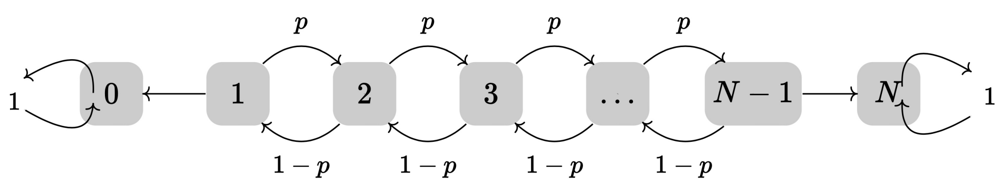
Theorem 3.7.12. *C a finite, closed communicating class $\implies$ all states in C positive recurrent*

3.7.2 Class properties

Type of Class	Finite	Infinite
Closed	positive recurrent	positive recurrent, null recurrent, transient
Not Closed	transient	transient

3.8 Application: The gambler's ruin problem

3.8.1 The problem and the results



Consider a gambler with initial fortune $i \in \{0, 1, \ldots, N\}$. At each play of the game, the gambler has

- probability p of winning one unit
- probability q of losing one unit
- each successive game is independent

What is the probability, a gambler starting at i units, has their fortune reach N before 0 ?

Let X_n denote gamblers fortune at time n. Then $\{X_n\}_{n \in \mathbb{N}_0}$ is a Markov Chain with transition probabilities, shown in diagram above.
This yields 3 communicating classes.

$$\underbrace{C_1 = \{0\}, C_2 = \{N\}}_{\substack{\text{positive recurrent} \\ \text{since finite and closed}}}, T_1 = \{1, 2, \ldots, N-1\}$$

Define the following for our problem:
Define first time X visits state i as

$$V_i = \min\{n \in \mathbb{N}_0 : X_n = i\}$$

$$h_i = h_i(N) = P(V_N < V_0 \mid X_0 = i)$$

This yields the following recurrence relation

$$h_i = h_{i+1}p + h_{i-1}q, \ i = 1, 2, \ldots, N-1$$

Theorem 3.8.1. *From above we achieve*

$$h_i = h_i(N) = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \frac{1}{2}; \\ \frac{i}{N}, & \text{if } p = \frac{1}{2}. \end{cases}$$

Theorem 3.8.2. *We also have*

$$\lim_{N \to \infty} h_i(N) = h_i(\infty) = \begin{cases} 1 - (q/p)^i, & \text{if } p > \frac{1}{2}; \\ 0, & \text{if } p \leq \frac{1}{2}. \end{cases}$$

- $p > \frac{1}{2} \implies \frac{q}{p} < 1 \implies \lim_{N \to \infty} (\frac{q}{p})^N = 0$
- $p < \frac{1}{2} \implies \frac{q}{p} > 1 \implies \lim_{N \to \infty} = \infty$

3.9 Stationarity

Definition 3.9.1. *(Distributions)*

1. *row vector* λ *a* ***distribution*** *on* E *if*

$$\forall j \in E, \lambda_j \geq 0, \quad \text{and} \sum_{j \in E} = 1$$

2. *row vector* λ *with non-negative entries is called* ***invariant*** *for transition matrix* P *if*

$$\lambda P = \lambda$$

3. *row vector* π *is* ***invariant/stationary/equilibrium distribution*** *of Markov chain on* E *with transition matrix* P *if*

   (a) π *a distribution*

   (b) *it is invariant*

$$\pi P^n = \pi$$

3.9.1 Stationarity distribution for irreducible Markov Chains

Theorem 3.9.2. *An irreducible chain has stationary distribution* $\pi \iff$ *all states are positive recurrent.* π *unique stationary distribution, s.t* $\pi_i = \mu_i^{-1} \forall i$

Lemma 3.9.3. *For markov chain* X *we have* $\forall j \in E, n, m \in \mathbb{N}$

$$f_{jj}(m+n) = \sum_{i \in E, i \neq j} l_{ji}(m) f_{ij}(n)$$

For $l_{ji}(n) = P(X_n = i.T_j \geq n \mid X_0 = j)$

Corollary 3.9.4. *For Markov Chain* X *we have* $\forall i, j \in E, i \neq j$ *and* $\forall n, m \in \mathbb{N}$

$$f_{jj}(m+n) \geq l_{ji}(m) f_{ij}(n)$$

Lemma 3.9.5. *Let* $i \neq j$ *Then* $l_{ji}(1) = p_{ji}$*, and for integers* $n \geq 2$

$$l_{ji}(n) = \sum_{r \in E : r \neq j} p_{ri} l_{jr}(n-1)$$

Lemma 3.9.6. $\forall j \in E$ *of an irreducible, recurrent chain, the vector* $\rho(j)$ *satisfies* $\rho_i(j) < \infty \ \forall i$ *and further* $\rho(j) = \rho(j)P$

Lemma 3.9.7. *Every irreducible, positive, recurrent chain has a stationary distribution*

Theorem 3.9.8. *If the chain is irreducible and recurrent, then* $\exists \mathbf{x} > 0$ *s.t* $\mathbf{x} = \mathbf{xP}$ *unique up to multiplicative constant.*

$$\text{Chain is} \begin{cases} \text{positive recurrent}, & \text{if } \sum_i x_i < \infty; \\ \text{null}, & \text{if } \sum_i x_i = \infty. \end{cases}$$

Lemma 3.9.9. *Let* T *a non-negative integer valued random variable on probability space* $(\Omega, \mathcal{F}, P)$*, with* $A \in \mathcal{F}$ *an event s.t* $P(A) > 0$*. Can show that*

$$E(T \mid A) = \sum_{n=1}^{\infty} P(T \geq n \mid A)$$

Theorem *(Dominated convergence theorem)*
Let $\mathcal{I}$ be a countable index set.
If $\sum_{i\in\mathcal{I}} a_i(n)$ is an absolutely convergent series $\forall n \in N$ s.t

1. $\forall i \in \mathcal{I}$ the limit $\lim_{n\to\infty} a_i(n) = a_i$ exists
2. $\exists$ seq. $(b_i)_{i\in I}$ s.t $b_i \geq 0\,\forall i$ and $\sum_{i\in\mathcal{I}} b_i < \infty$ s.t $\forall n, i : |a_i(n)| \leq b_i$

Then $\sum_{i\in\mathcal{I}} |a_i| < \infty$ and

$$\sum_{i\in I} a_i = \sum_{i\in I} \lim_{n\to\infty} a_i(n) = \lim_{n\to\infty} \sum_{i\in\mathcal{I}} a_i(n)$$

3.9.2 Limiting distribution

Definition 3.9.12. *A distribution π is the limiting distribution of a discrete-time Markov Chain if, $\forall i, j \in E$ we have*

$$\lim_{n\to\infty} p_{ij}(n) = \pi_j$$

Definition 3.9.14. *For irreducible aperiodic chain we have*

$$\lim_{n\to\infty} p_{ij}(n) = \frac{1}{\mu_j}$$

3.9.3 Ergodic Theorem

Theorem 3.9.16. *(Ergodic Theorem)*
Suppose we have irreducible Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ with state space E. Let μ_i the mean recurrence time to state $i \in E$

$$V_i(n) = \sum_{k=0}^{n-1} \mathbf{1}_{\{X_k=i\}}$$

The number of visits to i before n
So we have $V_i(n)/n$ the proportion of time before n spent at i

$$P\left(\frac{V_i(n)}{n} \to \frac{1}{\mu_i},\ \text{as } n \to \infty\right) = 1$$

Summary: Properties of irreducible Markov Chains
3 kinds of irreducible Markov Chains

1. **Positive recurrent**
   (a) Stationary distribution π exists
   (b) Stationary distribution is unique
   (c) All mean recurrence times are finite and $\mu_i = \frac{1}{\pi_i}$
   (d) $V_i(n)/n \xrightarrow[n\to\infty]{} \pi_i$
   (e) If chain aperiodic
   $$\lim_{n\to\infty} P(X_n = i) = \pi_i, \forall i \in E$$
2. **Null recurrent**
   (a) Recurrent, but all mean recurrence times are infinite
   (b) No stationary distribution exists
   (c) $V_i(n)/n \xrightarrow[n\to\infty]{} 0$

(d)

$$\lim_{n\to\infty} P(X_n = i) = 0, \forall i \in E$$

3. **Transient**
   (a) Any particular state is eventually never visited
   (b) No stationary distribution exists
   (c) $V_i(n)/n \xrightarrow[n\to\infty]{} 0$
   (d)

$$\lim_{n\to\infty} P(X_n = i) = 0, \forall i \in E$$

3.9.4 Properties of the elements of a stationary distribution associated with transient or null-recurrent states

Theorem 3.9.17. *Let X a time-homogeneous Markov Chain on countable state space E*
If π a stationary distribution of X, $i \in E$ either transient or null-recurrent, then $\pi_i = 0$

3.9.5 Existence of a stationary distribution on a finite state space

Theorem 3.9.19. *If state space finite $\implies \exists$ at least one positive recurrent communicating class*

Theorem 3.9.20. *Suppose finite state space. The stationary distribution π for transition matrix P unique $\iff$ there is a unique closed communicating class*

Corollary 3.9.21. *Markov chain on finite state space, and $N \geq 2$ closed classes.*
C_i the closed classes of Markov chain and $\pi^(i)$ the stationary distribution associated with class C_i using construction

$$\pi_j^{(i)} = \begin{cases} \pi_j^{C_i}, & \text{if } j \in C_i; \\ 0, & \text{if } j \notin C_i. \end{cases}$$

Then every stationary distribution of Markov Chain represented as

$$\sum_{i=1}^{N} \omega_i \pi^{(i)}$$

For weights $\omega_i \geq 0, \sum_{i=1}^{n} \omega_i = 1$

3.9.6 Limiting distributions on a finite state space

Theorem 3.9.23. *Let $K = |E| < \infty$ Suppose for some $i \in E$ that*

$$\lim_{n\to\infty} p_{ij}(n) = \pi_j, \quad \forall j \in E$$

Then π a stationary distribution

3.10 Time reversibility

Theorem 3.10.1. *For irreducible, positive recurrent Markov chain $\{X_n\}_{n \in 0,1,\ldots,N}, N \in \mathbb{N}$ assume π a stationary distribution, and P a transition matrix, and $\forall n \in \{0,1,\ldots,N\}$ the marginal distribution $\nu^{(n)} = \pi$*

$$Y_n = X_{N-n}, \quad \text{The reversed chain defined for } n \in \{0,1,\ldots,N\}$$

We have Y a Markov chain, satisfying

$$P(Y_{n+1} = j \mid Y_n = i) = \frac{\pi_j}{\pi_i} p_{ji}$$

Definition 3.10.2. *$X = \{X_n : n \in \{0, 1, \dots, N\}\}$ an irreducible Markov chain with stationary distribution π and marginal distributions $\nu^{(n)} = \pi,\ \forall n \in \{0, 1, \dots, N\}$*
*Markov chain X **time-reversible** if transition matrices of X and its reversal Y are the same.*

Theorem 3.10.3. *$\{X_n\}_{n\in\{0,1,\dots,N\}}$ time-reversible $\iff$, $\forall i, j \in E$*

$$\pi_i p_{ij} = \pi_j p_{ji}$$

Theorem 3.10.4. *For irreducible chain, if $\exists \pi$ s.t 3.10.1 holds $\forall i, j \in E$. Then the chain is time-reversible (once in its stationary regime) and positive recurrent with stationary distribution π*

4 Properties of the Exponential Distribution

4.1 Definition and basic properties

Definition 4.1.1. *(Exponential distribution)*
A continuous random variable X is $X \sim Exp(\lambda)$ if it has density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0; \\ 0, & \text{if otherwise.} \end{cases}$$

Cumulative distribution function

$$F_X(x) = \begin{cases} 0, & \text{if } x \le 0; \\ 1 - e^{-\lambda x}, & \text{if } x > 0. \end{cases}$$

Survival function of the exponential distribution is given by

$$P(X > x) = \begin{cases} 1, & \text{if } x \le 0; \\ e^{-\lambda x}, & \text{if } x > 0. \end{cases}$$

Theorem 4.1.2. *$X \sim Exp(\lambda)$ for $\lambda > 0$ Then*

1. *$E(X) = \frac{1}{\lambda}$*
2. *$\lambda X \sim Exp(1)$*

Theorem 4.1.3. *Let $n \in \mathbb{N}$ and $\lambda > 0$. Consider independent and identically distributed random variables $H_i \sim Exp(\lambda)$, for $i = 1, \dots, n$*
Let $J_n := \sum_{i=1}^{n} H_i$ Then J_n follows the Gamma(n, λ) distribution, i.e

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}$$

Theorem 4.1.4. *Let $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n$. Consider independent random variables $H_i \sim Exp(\lambda_i)$ for $i = 1, \dots, n$. Let $H := \min\{H_1, \dots, H_n\}$ Then*

1. *$H \sim Exp(\sum_{i=1}^{n} \lambda i)$*
2. *For any $k = 1, \dots, n, P(H = H_k) = \lambda_k / \sum_{i=1}^{n} \lambda_i$*

Theorem 4.1.5. *Consider a countable index set E and $\{H_i : i \in E\}$ independent random variables with $H_i \sim Exp(\lambda_i), \forall i \in E$. Suppose that $\sum_{i\in E} \lambda_i < \infty$ and set $H := \inf_{i\in E} H_i$*
Then the infimum is attained at a unique random value I of E with probability 1
H, I are independent, with $H \sim Exp(\sum_{i\in E} \lambda_i < \infty)$ and $P(I = i) = \lambda_i / \sum_{k\in E} \lambda_k$

Remark 4.1.6. *Suppose we have $X \sim Exp(\lambda_X), Y \sim Exp(\lambda_Y)$, Then*

$$P(X < Y) = P(\min\{X, Y\} = X) = \frac{\lambda_X}{\lambda_X + \lambda_Y}$$

4.2 Lack of memory property

Theorem 4.2.1. *(Lack of memory property)*
A continuous random variable $X : \Omega \to (0, \infty)$ has an exponential distribution $\iff$ has the lack of memory property

$$P(X > x + y \mid X > x) = P(X > y), \quad \forall x, y > 0$$

Remark 4.2.2. *A random variable $X : \Omega \to (0, \infty)$ has an exponential distribution $\iff$ has lack of memory property:*

$$P(X > x + y \mid X > x) = P(X > y), \quad \forall x, y > 0$$

4.3 Criterion for the convergence/divergence of an infinite sum of independent exponentially distributed random variables

Theorem 4.3.1. *Consider sequence of independent random variables $H_i \sim Exp(\lambda_i)$ for $0 < \lambda_i < \infty$ for all $i \in \mathbb{N}$ and let $J_\infty = \sum_{i=1}^{\infty} H_i$, Then:*

1. *If* $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty \implies P(J_\infty < \infty) = 1$
2. *If* $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty \implies P(J_\infty = \infty) = 1$

Lemma 4.3.2. *For $x \geq 1$, we have*

$$\log\left(1 + \frac{1}{x}\right) \geq \log(2)\frac{1}{x}$$

$$\log(1 + x) > \frac{x}{x+1}, \quad \textit{for } x > -1$$

5 Poisson Process

5.1 Remarks on continuous-time stochastic processes on a countable state space

5.3 Some Definitions

Definition 5.3.0. *A stochastic process $\{N_t\}_{t\geq 0}$ a **counting process** if N_t represents the total number of 'events' that have occurred up to time t*
Having the following properties:

1. $N_0 = 0$
2. $\forall t \geq 0, N_t \in \mathbb{N}_0$
3. *If* $0 \leq s \leq t, N_s \leq N_t$
4. *For $s < t, N_t - N_s =$ the number of events in interval $(s, t]$*
5. *Process is piecewise constant and has upward jumps of size 1 i.e $N_t - N_{t-} \in \{0, 1\}$*

Definition 5.3.1. *Let $(J_n)_{n\in\mathbb{N}_0}$ a strictly increasing sequence of positive random variables s.t $J_0 = 0$ almost surely.*
Define process $\{N_t\}_{t\geq 0}$ as

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{J_n \leq t\}},$$

Interpret J_n as the (random) time at which the nth event occurs.
The nth jump time.

5.3.1 Poisson Process: First Definition

Definition 5.3.0. *Define $o(\cdot)$ notation.*
A function f is $o(\delta)$ if

$$\lim_{\delta \downarrow 0} \frac{f(\delta)}{\delta} = 0$$

With the following properties

- *if f, g are $o(\delta)$ then so is $f + g$*
- *if f is $o(\delta)$ and $c \in \mathbb{R}$ then cf is $o(\delta)$*

Definition 5.3.3. *A **Poisson process** $\{N_t\}_{t\geq 0}$ of rate $\lambda > 0$ is a non-decreasing stochastic process with values in $\mathbb{N}_0$ satisfying:*

1. $N_0 = 0^1$
2. *Increments are independent, that is given any $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$ random variables $N_{t_0}, N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent*
3. *The increments are stationary, Given any 2 distinct times $0 \leq s < t, \forall k \in \mathbb{N}_0$*

$$P(N_t - N_s = k) = P(N_{t-s} = k)$$

4. *There is a 'single arrival', i.e $\forall t \geq 0, \delta > 0, d \to 0$:*

$$P(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta)$$
$$P(N_{t+\delta} - N_t \geq 2) = o(\delta)$$

5.3.2 Poisson Process: Second definition

Definition 5.3.4. *A **Poisson Process** $\{N_t\}_{t\geq 0}$ of rate $\lambda > 0$ is a stochastic process with values in $\mathbb{N}_0$ satisfying*

1. $N_0 = 0$
2. *Increments are independent, that is given any $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$ random variables $N_{t_0}, N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent*
3. *The increments are stationary, Given any 2 distinct times $0 \leq s < t, \forall k \in \mathbb{N}_0$*

$$P(N_t - N_s = k) = P(N_{t-s} = k)$$

4. $\forall t \geq 0, N_t \sim Poi(\lambda t)$

$$\forall k \in \mathbb{N}_0, P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

5.3.3 Right-continuous modification

Definition 5.3.0. *For 2 stochastic processes $\{X_t\}_{t\geq 0}, \{Y_t\}_{t\geq 0}$, say X a modification of Y if*

$$X_t = Y_t, \text{ almost surely for each} t \geq 0$$

$$P(X_t = Y_t) = 1, \forall t \geq 0$$

Can show that for each Poisson process, $\exists!$ modification which is càdlàg, (right continuous with left limits).

Remark 5.3.5. *Note that the jump chain of the Poisson Process given by $Z = (Z_n)_{n\in\mathbb{N}_0}$, where $Z_n = n, n \in \mathbb{N}_0$*

5.3.4 Equivalence of definitions

Theorem 5.3.6. *Definition 5.3.3, 5.3.4 are equivalent*

Lemma 5.3.7. *Laplace transform of a Poisson random variable of mean* $\lambda t, X \sim Poi(\lambda t)$ *for* $\lambda > 0, t > 0$ *is given by*

$$\mathcal{L}_X(u) = \exp\{\lambda t[e^{-u} - 1]\}, \quad \forall u > 0$$

5.4 Some properties of Poisson processes

5.4.1 Inter-arrival time distribution

Definition 5.4.1. *Let* $\{N_t\}_{t\geq 0}$ *a Poisson process of rate* $\lambda > 0$
Then the inter-arrival times are independently and identically distributed exponential random variables with parameter λ

5.4.2 Time to the n^{th} event

Theorem 5.4.2. *We have* $\forall n \in \mathbb{N}$, *the time to the* n^{th} *event* J_n *follows a Gamman,*λ *distribution, with density*

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \ t > 0$$

5.4.3 Poisson process: Third definition

Definition 5.4.4. *A* ***Poisson process*** $\{N_t\}_{t\geq 0}$ *of rate* $\lambda > 0$ *is a stochastic process with values in* $\mathbb{N}_0$ *s.t*

1. $H_1, H_2, \ldots$ *denote independently and identically exponentially distributed random variables with parameter* $\lambda > 0$
2. *Let* $J_0 = 0$ *and* $J_n = \sum_{i=1}^n H_i$
3. *Define*

$$N_t = \sup\{n \in \mathbb{N}_0 : J_n \leq t\}, \quad \forall t \geq 0$$

Theorem 5.4.5. *Definitions 5.3.3, 5.3.4, 5.4.4 are equivalent*

5.4.4 Conditional distribution of the arrival times

Theorem 5.4.6. *Let* $\{N_t\}_{t\geq 0}$ *be a Poisson process of rate* $l > 0$. *Then* $\forall n \in \mathbb{N}, t > 0$, *the conditional density of* $\big(J_1, \ldots, J_n\big)$ *given by* $N_t = n$ *is given by*

$$f_{\big(J_1, \ldots, J_n\big)}\big(t_1, \ldots, t_n | N_t = n\big) = \begin{cases} \frac{n!}{t^n}, & \text{if } 0 < t_1 < \ldots < t_n \leq t; \\ 0, & \text{otherwise .} \end{cases}$$

Remark 5.4.7. *The above theorem says, conditional on the fact n events have occurred in* $[0, t]$, *the times* $\big(J_1, \ldots, J_n\big)$ *at which the events occur, when considered as unordered random variables are independently and uniformly distributed on* $[0, t]$

5.5 Some extensions to Poisson processes

5.5.1 Superposition

Theorem 5.5.2. *Given n independent Poisson processes* $\{N_t^(1)\}_{t\geq 0}, \ldots, \{N_t^{(n)}\}_{t\geq 0}$ *with respective rates,* $\lambda_1, \ldots, \lambda_n > 0$ *define*

$$N_t = \sum_{i=1}^n N_t^{(i)}, \quad t \geq 0$$

Then $\{N_t\}_{t\geq 0}$ *a Poisson process with rate* $\lambda = \sum_{i=1}^n \lambda_i$ *and is called a* ***superposition of Poisson processes***

5.5.2 Thinning

Theorem 5.5.5. *Let $\{N_t\}_{t\geq 0}$ a Poisson process with rate $\lambda > 0$. Assume that each arrival, independent of other arrivals, is marked as a type k event with probability p_k for $k = 1, \dots, n$ where $\sum_{i=1}^{n} p_i = 1$.*
Let $N_t^{(k)}$ denote the number of type k events in $[0, t]$. Then $\{N_t^{(k)}\}_{t\geq 0}$ a Poisson process with rate λp_k and the processes

$$\{N_t^(1)\}_{t\geq 0}, \dots, \{N_t^{(n)}\}_{t\geq 0}$$

*are independent. Each process called a **thinned Poisson process***

5.5.3 Non-homogeneous Poisson processes

Definition 5.5.6. *Let $\lambda : [0, \infty) \mapsto (0, \infty)$ denote a non-negative and locally integrable function, called the **intensity function***
*A non-decreasing stochastic process $N = \{N_t\}_{t\geq 0}$ with values in $\mathbb{N}_0$ called a **non-homogeneous Poisson process** with intensity function $(\lambda(t))_{t\geq 0}$ if it satisfies the following:*

1. $N_0 = 0$
2. *N has independent increments*
3. *'Single arrival' property, For $t \geq 0, \delta > 0$*

$$P(N_{t+\delta} - N_t = 1) = \lambda(t)\delta + o(\delta)$$
$$P(N_{t+\delta} - N_t \geq 2) = o(\delta)$$

Note that (3) also implies that

$$P(N_{t+\delta} - N_t = 0) = 1 - \lambda(t) + o(\delta)$$

Theorem 5.5.7. *Let $N = \{N_t\}_{t\geq 0}$ denote a non-homogeneous Poisson process with continuous intensity function $(\lambda(t))_{t\geq 0}$ Then*

$$N_t \sim Poi(m(t)), \quad where \quad m(t) = \int_0^t \lambda(s) ds$$

i.e. $\forall t \geq 0, n \in \mathbb{N}_0$

$$P(N_t = n) = \frac{[m(t)]^n}{n!} e^{-m(t)}$$

5.5.4 Compound Poisson processes

Definition 5.5.12. *Let $\{N_t\}_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$.*
$Y_1, Y_2, \dots$ be a sequence of independent and identically distributed random variables, that are independent of $\{N_t\}_{t\geq 0}$. Then the process $\{S_t\}_{t\geq 0}$ with

$$S_t = \sum_{i=1}^{N_i} Y_i, \quad t \geq 0$$

*is a **compound Poisson process***

Theorem 5.5.13. *Let $\{S_t\}_{t\geq 0}$ a compound Poisson process. Then for $t \geq 0$*

$$E(S_t) = \lambda t E(Y_1), \quad Var(S_t) = \lambda t E(Y_1^2)$$

as defined in Definition 5.5.12

5.6 The Cramér-Lundberg model in insurance mathematics

Definition 5.6.1. *The* ***Cramér-Lundberg model*** *is given by the following five conditions.*

1. *Claim size process is denoted by* $Y = (Y_k)_{k\in\mathbb{N}}$*, for* Y_k *denoting the positive i.i.d random variables with finite mean* $\mu = E(Y)1$ *and variance* $\sigma^2 = Var(Y_1) \leq \infty$

2. *Claim times occur at the random instants of time*

$$0 < J_1 < J_2 < \ldots a.s..$$

3. *The claim arrival process is denoted by*

$$N_t = \sup\{n \in \mathbb{N} : J_n \leq t\}, t \geq 0$$

which is the number of claims in the interval $[0, t]$.

4. *The inter-arrival times are denoted by*

$$H_1 = J_1, H_k = J_k - J_{k-1}, k = 2, 3, \ldots$$

and are independent and exponentially distributed with parameter λ

5. *sequences* $(Y_k, (H_k))$ *are independent of each other*

Definition 5.6.3. *The* ***Total claim amount*** *is defined as the process* $(S_t)_{t\geq 0}$ *satisfying*

$$S_t = \begin{cases} \sum_{i=1}^{N_t} Y_i, & \text{if } N_t > 0; \\ 0, & \text{if } N_t = 0. \end{cases}$$

Observe that the total claim amount is modelled as a compound Poisson process.

Theorem 5.6.4. *The total claim amount distribution given by*

$$P(S_t \leq x) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} P\left(\sum_{i=1}^{n} Y_i \leq x\right), \quad x \geq 0, t \geq 0$$

and $P(S_t \leq x) = 0$ *for* $x < 0$

Definition 5.6.5. *The* ***risk process*** $\{U_t\}_{t\geq 0}$ *is defined as*

$$U_t = u + ct - S_t, \quad t \geq 0$$

where $u \geq 0$*, the* ***initial capital*** *and* $c > 0$ *denotes the* ***premium income rate***

Definition 5.6.7. *We have the following definitions*

1. *The* ***ruin probability in finite time*** *is given by*

$$\psi(u, T) = P(U_t < 0 \ \text{ for some } t \leq T), \ 0 < T < \infty, u \geq 0$$

2. *The* ***ruin probability in infinite time*** *is given by*

$$\psi(u) := \psi(u, \infty), u \geq 0$$

Theorem 5.6.8.

$$E(U_t) = u + ct - \lambda t\mu + (c - \lambda\mu)t$$

A minimal requirement for choosing the premium could be

$$c > \lambda\mu$$

referred to as the ***net profit condition***

5.7 The Coalescent Process

5.7.1 Problem

- Given collection of n individuals - observe a DNA sequence from the individual
- A DNA sequence a collection of letters; A,C,T and G - for simplicity take that only one letter observed
- Coalescent process provides genealogical tree representation of this data. A tree-like structure representing the history of the individuals backward in time.
  Individuals coalesce until we have only individual - the most recent common ancestor.

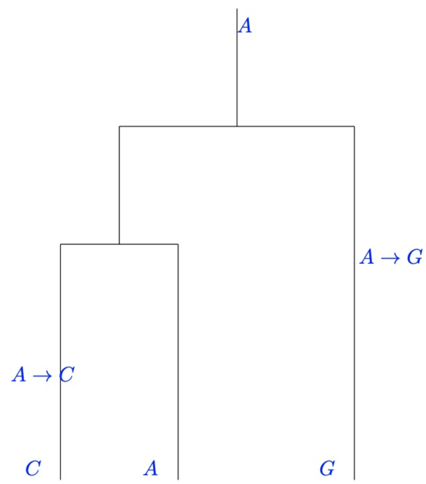


Figure 1: A Coalescent Graph

5.7.2 The Process

- At start of process we have $n \geq 2$ individuals (all of the same DNA base)
- Each pair of individuals coalesce according to an (independent) Poisson process of rate 1
- We have $\binom{n}{2}$ pairs - time to first coalescent event is exponential random variable of rate $\binom{n}{2}$ - since we consider the minimum of $\binom{n}{2}$ independent $Exp(1)$-distributed random variables.
- At first event - 2 individuals picked uniformly at random and combined
- Continue this until there is only one individual - the most recent common ancestor
- So we have $n - 1$ coalescent events
- Model, assumes all individuals have the same DNA base, so we require another mechanism - a mutation process
- In this process the number of individuals decrease - our first example of a death process.

5.7.3 Time to most recent common ancestor

Time to most recent common ancestor estimated i.e. the height of the tree, estimated by

$$E\left(\sum_{k=1}^{n-1} H_k\right), \text{ for } n \in \mathbb{N}, n \geq 2$$

Where we have that H_k the time to k^{th} coalescence

$$H_k \sim \text{Exp}\left(\binom{n-(k-1)}{2}\right) \implies E(H_k) = \left(\binom{n-(k-1)}{2}\right)^{-1}$$

So we have that

$$\begin{aligned}
E(\sum_{k=1}^{n-1} H_k) &= \sum_{k=1}^{n-1} E(H_k) \\
&= \sum_{k=1}^{n-1} \left(\frac{(n-k+1)!}{(n-k-1)!2!}\right)^{-1} = \sum_{k=1}^{n-1} \frac{2(n-k-1)!}{(n-k+1)!} \\
&= \sum_{k=1}^{n-1} \frac{2}{(n-k+1)(n-k)} = \sum_{k=1}^{n-1} \frac{2}{k(k+1)} \\
&= 2\left(1 - \frac{1}{n}\right)
\end{aligned}$$

Further, since $H_{n-1} \sim \text{Exp}\left(\binom{2}{2}\right)$

$$E(H_{n-1}) = 1$$

6 Continuous-time Markov Chains

6.1 Some definitions

Definition 6.1.1. *A continuous-time process $\{X_t\}_{t\in[0,\infty)}$ satisfies the **Markov property** if*

$$P(X_{t_n} = j \mid X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}) = P(X_{t_n} = j \mid X_{t_{n-1}} = i_{n-1})$$

for all $j, i_1, \ldots, i_{n-1} \in E$ and for any sequence $0 \leq t_1 < \ldots < t_n < \infty$ of times (with $n \in \mathbb{N}$)

Definition 6.1.2. *The **transition probability** $p_{ij}(s,t)$ is, for $s \leq t, i, j \in E$*

$$p_{ij}(s,t) = P(X_t = j \mid X_s = i)$$

*also, the chain is **homogeneous** if*

$$p_{ij}(s,t) = p_{ij}(0, t-s)$$

Write $p_{ij}(t-s) = p_{ij}(s,t)$ in this case
Let $\mathbf{P}_t = (p_{ij}(t))$

Theorem 6.1.3. *The family $\{\mathbf{P}_t : t \geq 0\}$ is a **stochastic semigroup**; that is, it satisfies*

1. $\mathbf{P}_0 = I_{K\times K}$
2. $\mathbf{P}_t$ *is stochastic - non-negative entries with rows summing to* 1
3. *The Chapman-Kolmogorov equations hold:*

$$\mathbf{P}_{s+t} = \mathbf{P}_s\mathbf{P}_t, \quad \forall s, t \geq 0$$

Definition 6.1.4. *The semigroup $\{\mathbf{P}_t\}$ is called **standard** if*

$$\lim_{t\downarrow 0} \mathbf{P}_t = \mathbf{I}\ (= \mathbf{P}_0)$$

where $\mathbf{I} = \mathbf{I}_{K\times K}$
A semigroup standard $\iff$ its elements $p_{ij}(t)$ are continuous functions in t

6.2 Holding times and alarm clocks

6.2.1 Holding times

Suppose that we have $\{X_t\}_{t\geq 0}$ a continuous-time homogeneous Markov Chain, suppose that $t \geq 0$ and for, $i \in E$, we have $X_t = i$. Given $X_t = i$, define

$$H_{|i} = \inf\{s \geq 0 : X_{t+s} \neq i\}$$

to be the **holding time at state** i , that is the length of time that a continuous-time Markov chain started in state i stays in state i before transitioning to a new state.
Note that holding times does not depend on t since we work under time-homogeneity assumption

$$\inf\{s \geq 0 : X_{t+s} \neq i\} \mid X_t = i \overset{\text{def.}}{=} \inf\{s \geq 0 : X_s \neq i\} \mid X_0 = i$$

Theorem 6.2.2. *The holding times $H_{|i}$, for $i \in E$ follows an exponential distribution*

6.2.2 Describing the evolution of a Markov Chain using exponential holding times

Can describe the evolution of continuous-time Markov chains by specifying **transition rates** between states and using the concept of **exponential alarm clocks**

- $\forall i \in E$ denote n_i- number of states which can be reached from state i
- Associate n_i independent, exponential alarm clocks with rates q_{ij} provided j can be reached from state i
- When chain first visits state i, all n_i exponential alarm clocks are set simultaneously
- First alarm clock which rings, determines which state the chain transitions to.
- As soon as state j has been reached - set n_j independent exponential alarm clocks associated to $j$and repeat the process

q_{ij} **- transition rates**

- Let $i \neq j$, with $q_{ij} > 0$ denote the transition rates when state j can be reached from state i
- Let $i \neq j$, set $q_{ij} = 0$ if j can't be reached from i
- Also set $q_{ii} = 0, \forall i \in E$
- The minimum/infimum of the n_i exponential alarm clocks of state i,follows an exponential distribution with rate

  $$q_i = \sum_{j\in E} q_{ij}$$

- $P(i \to j) = P(q_i = q_{ij}) = \frac{q_{ij}}{q_i}$
- Hence, the transition probabilities of embedded chain Z given by

  $$p_{ij}^Z = \frac{q_{ij}}{q_i}$$

  We assumed above that $0 < q_i < \infty$. In case that $q_i = 0$ then we have $p_{ii}^Z = 1$

6.3 The generator

Definition 6.3.1. *The generator* $\mathbf{G} = (g_{ij})_{i,j\in E}$ *of the Markov chain with stochastic semigroup* $\mathbf{P}_t$ *is defined as the* $card(E) \times card(E)$ *matrix given by*

$$\mathbf{G} := \lim_{\delta\downarrow 0} \frac{1}{\delta}[\mathbf{P}_\delta - \mathbf{I}] = \lim_{\delta\downarrow 0} \frac{1}{\delta}[\mathbf{P} - \mathbf{P}_0]$$

That is, $\mathbf{P}_t$ *differentiable at* $t = 0$

6.3.1 Transition probabilities of the associated jump chain

Can now derive the transition probabilities of the embedded/jump chain - expressing them in terms of the generator
if $X_t = i$- stay at i for exponentially distributed time with rate $-g_{ii} = q_i$ and then moves to other state j
Probability that the chain jumps to $j \neq i$ is $-g_{ij}/g_{ii}$
i.e for $i \neq j$,

$$p^Z_{ij} = -\frac{g_{ij}}{g_{ii}} = \frac{q_{ij}}{q_i}$$

Equivalent to

$$q_{ij} = q_i p^Z_{ij}$$

6.4 The forward and backward equations

Theorem 6.4.1. *Subject to regularity conditions, a continuous-time Markov chain with stochastic semigroup* $\{\mathbf{P}_t\}$ *and generator* $\mathbf{G}$ *satisfies the* ***Kolmogorov forward equation***

$$\mathbf{P}'_t = \mathbf{P}_t\mathbf{G}$$

and the ***Kolmogorov backward equation***

$$\mathbf{P}'_t = \mathbf{G}\mathbf{P}_t, \quad \forall t \geq 0$$

6.4.1 Matrix exponentials

6.5 Irreducibility, stationarity and limiting distribution

Definition 6.5.1. *Chains is* ***irreducible*** *if for any* $i, j \in E$ *we have* $p_{ij}(t) > 0$, *for some* t

Theorem 6.5.2. *If* $p_{ij}(t) > 0$, *for some* $t > 0$ *then* $p_{ij}(t) > 0$, $\forall t > 0$

Definition 6.5.3. *A distribution* π *is the* ***limiting distribution*** *of a continuous-time Markov chain if, for all states* $i, j \in E$ *we have*

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j$$

Definition 6.5.4. *A distribution* π *is a stationary distribution if* $\pi = \pi\mathbf{P_t}$ $\forall t \geq 0$

Theorem 6.5.5. *Subject to regularity conditions, we have* $\pi = \pi\mathbf{P_t}, \forall t \geq 0 \iff \pi\mathbf{G} = 0$

Theorem 6.5.6. *Let* X *an irreducible Markov chain with a standard semigroup* $\{\mathbf{P}_t\}$ *of transition probabilities*

1. *If* $\exists$ *stationary distribution* π *then it is unique and* $\forall i, j \in E$

$$\lim_{t\to+\infty} p_{ij}(t) = \pi_j$$

2. *If there is no stationary distribution then*

$$\lim_{t\to+\infty} p_{ij}(t) = 0 \ \forall i, j \in E$$

6.6 Jump chain and explosion

Subject to regularity conditions, can construct the jump chain Z from a continuous time Markov chain X as follows

- J_n denote the nth change in value of the chain X and set $J_0 = 0$
- Values $Z_n = X_{J_n+}$ of X form a discrete-time Markov Chain $Z = \{Z_n\}_{n\in\mathbb{N}_0}$
- Transition matrix of Z denoted by $\mathbf{P}^Z$ and satisfies
  - $p_{ij}^Z = g_{ij}/g_i$ if $g_i := -g_{ii} > 0$
  - if $g_i = 0$, then the chain gets absorbed in state i once it gets there for the first time.
- If $Z_n = j$ then the holding time $H_{n+1} = J_{n+1} - J_n = H_{|j}$ has exponential distribution with parameter g_j
- The chain Z is called the **jump chain of** X

Consider the converse - a discrete-time Markov chain Z taking values in E - Try and find a continuous-time Markov chain X having Z as its jump chain - Many such X exist

- Let $\mathbf{P}^Z$ denote transition matrix of the discrete-time Markov chain Z taking values in E Assume $p_{ii}^Z = 0, \ \forall i \in E$
- $i \in E$ let g_i denote non-negative constants. Define

$$g_{ij} = \begin{cases} g_i p_{ij}^Z, & \text{if } i \neq j; \\ -g_i, & \text{if } i = j. \end{cases}$$

  Construction of continuous-time Markov chain X done as follows

- Set $X_0 = Z_0$
- After holding time $H_1 = H_{|Z_0} \sim Exp(g_{Z_0})$ the process jumps to state Z_1
- After holding time $H_2 = H_{|Z_1} \sim Exp(gZ_1)$ the process jumps to state Z_3
- Formally: conditionally on the values Z_n of chain Z let $H_1, H_2, \ldots$ be independent random variables with exponential distribution $H_i \sim Exp(gZ_{i-1})$, $i = 1, 2, \ldots$. Set $J_n = H_1 + \ldots + H_n$
- Then define

$$X_t = \begin{cases} Z_n, & \text{if } J_n \leq t \leq J_{n+1} \text{ for some } n; \\ \infty, & \text{otherwise i.e. if } J_\infty \leq t. \end{cases}$$

- Note that the special state ∞ added in case the chain explodes Recall that $J_\infty = \lim_{n\to\infty} J_n \cdot J_\infty$ called the **explosion time** say chain **explodes** if

$$P(J_\infty < \infty) > 0$$

  Can show that

- X a continuous-time Markov chain with state space $E \cup \{\infty\}$
- Matrix G is the generator of X
- Z is the jump chain of X

Theorem 6.6.1. *The cain X constructed above does not explodes if any of the following conditions hold*

1. *State space E is finite*
2. $\sup_{i\in E} g_i < \infty$
3. *$X_0 = i$ where i a recurrent state for the jump chain Z*

6.7 Birth processes

Definition 6.7.1. *A birth process with intensities $\lambda_0, \lambda_1, \ldots \geq 0$ a stochastic process $\{N_t\}_{t\geq 0}$ with values in $\mathbb{N}_0$, such that*

1. *Non-decreasing process: $N_0 \geq 0$; if $s < t$ then $N_s \leq N_t$*
2. *There is a 'single arrival' i.e. the infinitesimal transition probabilities are for $t \geq 0, \delta > 0, n, m \in \mathbb{N}_0$*

$$P(N_{t+\delta} = n + m \mid N_t = n) = \begin{cases} 1 - \lambda_n \delta + o(\delta), & \text{if } m = 0; \\ \lambda_n \delta + o(\delta), & \text{if } m = 1; \\ o(\delta) & \text{if } m > 1 \end{cases}$$

3. *Conditionally independent increments: Let $s < t$ then the conditional on the values of N_s, the increment $N_t - N_s$ is independent of all arrivals prior to s*

Note that by conditionally independent increments, mean that for $0 \leq s < t$, conditional on the value of N_s, the increment $N_t - N_s$ independent of all arrivals prior to s i.e. for $k, l, x(r) \in \{0, 1, 2, \ldots\}$ for $0 \leq r < s$ we have

$$P(N_t - N_s = k \mid N_s = l, N_r = x(r) \text{ for } 0 \leq r < s) = P(N_t - N_s = k \mid N_s = l)$$

Birth process a continuous-time Markov chain
A Poisson process a special case of a birth process (with $\lambda_n = \lambda, \forall n \in \mathbb{N}_0$)
With the general case, birth rates depend on the current state of the process.

6.7.1 The forward and backward equations

Let $\{N_t\}$ a birth process with positive intensities $\lambda_0, \ldots$
With transition probabilities

$$p_{ij}(t) = P(N_{t+s} = j \mid N_s = i) = P(N_t = j \mid N_0 = i), \quad \text{for } i, j \in E$$

Theorem 6.7.5. *For $i, j \in E, i < j, t \geq 0$ the forward equations of a birth process are given by*

$$\frac{dp_{ij}(t)}{dt} = -\lambda_j p_{ij}(t) + \lambda_{j-1} p_{i,j-1}(t)$$

with $\lambda_{-1} = 0$, and the backward equation given by

$$\frac{dp_{ij}(t)}{dt} = -\lambda_i p_{ij}(t) + \lambda_i + p_{i+1,j}(t)$$

Where for both, boundary condition given by $p_{ij}(0) = \delta_{ij}$ - δ the Kronecker delta

Theorem 6.7.6. *Let $\{N_t\}_{t\geq 0}$ a birth process of positive intensities $\lambda_0, \ldots$ Then the forward equations have unique solutions which satisfies the backward equations*

6.7.2 Explosion of a birth process

Definition 6.7.7. *Let $J_0, J_1, \ldots$ denote the jump times of a birth process N*

$$J_0 = 0 \quad J_{n+1} = \inf\{t \geq J_n : N_t \neq N_{J_n}\}, \quad n \in \mathbb{N}_0$$

Further let $H_1, H_2, \ldots$ denote the corresponding holding times. As before, we write

$$J_\infty = \lim_{n\to\infty} J_n = \sum_{i=1}^{\infty} H_i$$

Then we say that explosion of the birth process N is possible if

$$P(J_\infty < \infty) > 0$$

Theorem 6.7.8. *Let N be a birth process started from $k \in \mathbb{N}_0$, with rates $\lambda_k, \lambda_{k+1}, \ldots > 0$ Then:*

$$\text{If } \sum_{i=k}^{\infty} \frac{1}{\lambda_i} \begin{cases} < \infty, & \text{Then } P(J_\infty < \infty) = 1 \text{ (Explosion occurs with probability 1)}; \\ = \infty, & \text{Then } P(J_\infty = \infty) = 1 \text{ (Probability explosion occurs is 0)}. \end{cases}$$

6.8 Birth-death processes

Definition 6.8.1. ***Birth-death process***
Suppose we are given the following process $\{X_t\}_{t\geq 0}$

1. $\{X_t\}_{t\geq 0}$ *is Markov chain on* $E = \mathbb{N}_0$
2. *The infinitesimal transition probabilities are (for* $t \geq 0, \delta > 0, n \in \mathbb{N}_0, m \in \mathbb{Z}$ *)*

$$P(X_{t+\delta} = n + m \mid X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta), & \text{if } m = 0; \\ \lambda_n + o(\delta), & \text{if } m = 1. \\ \mu_n \delta + o(\delta), & \text{if } m = -1 \\ o(\delta), & \text{if } |m| > 1 \end{cases}$$

3. *The birth rates* $\lambda_0, \lambda_1, \ldots$ *and the death rates* $\mu_0, \mu_1, \ldots$ *satisfy*

$$\lambda_i \geq 0, \quad \mu_i \geq 0 \quad \mu_0 = 0$$

We have the generator given by

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We take a look at the asymptotic behaviour of the process.
Suppose that $\mu_i, \lambda_i > 0, \forall i$*, where the rates make sense. Then using the claim* $\pi G = 0$

$$-\lambda_0 \pi_0 + \mu_1 \pi_1 = 0$$
$$\lambda_{n-1}\pi_{n-1} - (\lambda_n + \mu_n)\pi_n + \mu_{n+1}\pi_{n+1} = 0, \quad n \geq 1$$

7 Brownian Motion

7.2 From random walk to Brownian motion

7.2.1 Modes of convergence in distribution, Slutsky's theorem and the CLT

Definition 7.2.2. *(Convergence in probability)*
A sequence of random variables $X_1, X_2, \ldots$ *converges in probability to* X *written* $X_n \xrightarrow{P} X$ *if for each* $\epsilon > 0$

$$\lim_{n\to\infty} P(\{\omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}) = \lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

Definition 7.2.3. *(Convergence in distribution)*
Let the cumulative distribution function of X_n *and* X *be denoted by* F_n *and* F *respectively*
Say X_n *converges in distribution/weakly to* X*, written* $X_n \xrightarrow{D} X$ *if*

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{for every continuity point } x of F(x)$$

Theorem 7.2.4. *(Slutsky's theorem)*
Suppose that $X_n \xrightarrow{d} X, A_n \xrightarrow{P} a, B_n \xrightarrow{P} b$*, where* a, b *are (deterministic) constants. Then*

$$A_n X_n + B_n \xrightarrow{d} aX + b$$

Theorem 7.2.5. *(Central limit theorem)*
Let $Z_1, Z_2, \dots$ a sequence of i.i.d random variables of finite mean μ and finite variance σ^2. Then the distribution of

$$\frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} Z_i - n\mu\right)$$

tends to standard normal distribution as $n \to \infty$

7.3 Brownian Motion

Definition 7.3.1. *A real-valued stochastic process $B = \{B_t\}_{t\geq 0}$ a* ***standard Brownian motion*** *if*

1. *$B_0 = 0$ almost surely*
2. *B has independent increments*
3. *B has stationary increments*
4. *The increments are Gaussian, for $0 \leq s < t$*

$$B_t - B_s \sim N(0, (t-s));$$

5. *The sample paths are almost surely continuous i.e. the function $t \mapsto B_t$ almost surely continuous in t*

Definition 7.3.2. *Let $B = \{B_t\}_{t\geq 0}$ denote a standard Brownian motion.*
Stochastic process $Y = \{Y_t\}_{t\geq 0}$ defined by

$$Y_t = \sigma B_t + \mu_t, \forall t \geq 0$$

is called a ***Brownian motion with drift parameter*** *$\mu \in \mathbb{R}$ and* ***variance parameter*** *$\sigma^2, \sigma > 0$*
Note for $0 \leq s < t, Y_t - Y_s \sim N(\mu(t-s), \sigma^2(t-s))$

7.5 Finite dimensional distributions and transition densities

Theorem 7.5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ a continuous function satisfying some additional regularity conditions. Then the unique (continuous) solution $u_t(x)$ to the initial value problem*

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u_t(x)$$
$$u_0(x) = f(x)$$

is given by

$$u_t(x) = E[f(W_t^x)] = \int_{-\infty}^{\infty} p_t(x, y) f(y)\, \mathrm{d}y$$

where $\{W_t^x\}$ is a Brownian motion started at x

7.6 Symmetries and scaling laws

Proposition 7.6.1. *Let $\{B_t\}_{t\geq 0}$ a standard Brownian motion. Then each of the following processes is also a standard Brownian motion*

$\{-B_t\}_{t\geq 0}$		Reflection
$\{B_{t+s} - B_s\}_{t\geq 0}$	for $s \geq 0$	Translation
$\{aB_{t/a^2}\}_{t\geq 0}$	for $a \geq 0$	Rescaling **(Brownian scaling property)**
$\{tB_{1/t}\}_{t\geq 0}$		Inversion

7.6.1 Some remarks

First look at maximum and minimum processes

$$M_t^+ := \max\{B_s : 0 \leq s \leq t\}$$
$$M_t^- := \min\{B_s : 0 \leq s \leq t\}$$

These are well-defined, because the Brownian motion has continuous paths, and continuous functions always attain their maximal and minimal values on compact intervals
Observe that if the path B_t is replaced by its reflection $-B_t$ then the maximum and the minimum are interchanged and negated
Since $-B_t$ a Brownian motion, follows that $M_t^+, -M_t^-$ have same distribution

$$M_t^+ \stackrel{d}{=} -M_t^-$$

As a first example, consider implications of Brownian scaling property for the distributions of the maximum random variables M_t^+ . Fix $a > 0$, and define

$$\begin{aligned} B_t^* &:= aB_{t/a^2} \\ M_t^{+,*} &:= \max_{0\leq s\leq t} B_s^* \\ &= aM_{t/a^2}^+ \end{aligned}$$

By Brownian scaling property B_t^* is a standard Brownian motion, and so random variable M_t^{+*} has same distribution as M_{t^+}. Therefore

$$M_t^+ \stackrel{d}{=} aM_{t/a^2}^+$$

Can be shown, above implies that the sample paths of a Brownian motion are with probability one, nowhere differentiable

7.7 The reflection property and first-passage times

Proposition 7.7.1. *Let $x > 0$ then*

$$P(M_t^+ \geq x) = 2P(B_t > x) = 2 - \Phi(x/\sqrt{t})$$

Where Φ the normal c.d.f

7.8 A model for asset prices

A model for describing movement of an asset price $\{S_t\}_{0\leq t\leq T}, S_t \in \mathbb{R}^+$ is as follows

$$S_t = S_0 \exp\{(\mu - \sigma^2/2)t + \sigma B_t\}$$

where S_0 the initial value of the underlying stock. $\mu \in \mathbb{R}$ is the risk-free interest rate and σ the volatitily (the instantaneous standard deviation of the stock)
This process known as **geometric Brownian motion**.

It is well-known that this model does not fit the stylized features of financial returns data.
Real financial data does not follow the dynamic above; because in practice volatility of asset prices is typically not constant, and often responds to a variety of market conditions
We typically observe time-varying volatility clusters.

This has yielded much academic and industrial research into cases (which goes back to at least the late 1970s) where σ is a stochastic process, e.g:

$$S_t = S_o \exp\left\{\left(\mu t - \frac{1}{2}\int_0^t \sigma_s^2 \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s\right)\right\}$$
$$\sigma_t = \sigma_0 \exp\{\gamma t + \eta W_t\}$$

where W_t is an independent Brownian motion; such a model is termed a stochastic volatility model