MATH60013 – Mathematics of Business and Economics

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Spring 2023

Introduction

This course aims to provide a broad mathematical introduction to economics and its application in a business setting. In this introduction, I provide some initial definitions relevant to the content of the course, in addition to detailing the course structure and specifying some course objectives.

What is an economy?

An economy is an ecosystem, in which governments, markets, firms and individual consumers all interact with the aim of enabling the provision of goods and services in return for payment. Broadly speaking, there are four groups of agents that enable an economy to function:

- **individuals and households**, who act as consumers in obtaining goods and services from producers, and as suppliers in providing labour to producers;

- **firms**, who provide goods and services to consumers, and who also employ individuals from the first group; firms also act as consumers for other firms;

- **governments and regulatory bodies**, who provide oversight, regulation and intervention in order that their economies function smoothly and in service of particular goals;

- and **trade partners external to the economy**, who also interact with these agents to influence production and consumption within the economy.

The above definition of an economy indicates that economic analysis can be carried out at a number of different scales. Households and firms come together to trade goods and services within (conceptual) markets; the analysis of such interactions and the behaviour of these markets is the principal focus of **microeconomic theory**. In contrast, the actions of governments that influence the operation of markets, as well as the interaction of markets with external trading partners, falls under the heading of **macroeconomic analysis**.

This course will focus on the interaction of economic agents within a market economy, i.e. one where production and trade are private enterprises. As such, we consider an ecosystem whose constituent parts are each controlled by their own decision-making processes; the study of economics is therefore important to understand how these processes affect one another and how each agent should operate in order to satisfy their particular goals whilst acknowledging the behaviour of other agents and changes to the economic environment in general.

Course Structure

The structure of this course reflects this division of analyses. In Parts 1 and 2, we focus on microeconomics, before taking a macroeconomic viewpoint in Part 3.

- In **Part 1**, we will analyse the aims and objectives of both firms and consumers, and we will use mathematical arguments to show how these objectives lead to the observed behaviour in a competitive market environment.
- In **Part 2**, we will consider how the behaviour of firms and consumers is affected by the properties of the market in which they operate, and how their behaviour is affected by changes to the market environment.
- In **Part 3**, we analyse the macroeconomic environment; in particular, we explore aggregated concepts of supply and demand, we look at the circular flow of income and discuss the Gross Domestic Product (GDP).

Syllabus:

Theory of the firm
Profit maximisation for a competitive firm
Cost minimisation. Geometry of costs
Profit maximisation for a non-competitive firm

Theory of the consumer
Consumer preferences and utility maximisation
The Slutsky's equation

Levels of competition in a market
Consumers' and Producers' surplus
Deadweight loss

Macroeconomic theory
Circular flow of income
Gross Domestic Product
Social welfare and allocation of income

Mathematical Methods:
(Constraint) Optimisation. Quasi-concavity. Preferences relations and orders.

Course Objectives

This course provides an introduction to the fundamental aspects of both microeconomics and macroeconomics, using a mostly rigorous mathematical approach to both the exposition and demonstration of these subjects.
In a business context, this course will provide you with the tools required to analyse the goals of a firm and the decisions that a firm may make in the context of their particular market. At the end of the course, you should understand the effects that these decisions

have on the firm itself, on the various connected individuals and firms, and on the economy as a whole.

Solving problems in this course will require both an economic understanding of the concepts as well as a sound mathematical derivation. That means that you should be prepared to come up with mathematical proofs as well as to explain notions in form of (very) short essays.

Additional Course Information

Lectures: Three lectures/week: Monday, 3pm, Wednesday, 11am, and Friday, 9am (check your calendars for room)

Office Hour: Friday, 11am, starting on January 13

Problem Classes: Bi-weekly; starting on January 20

The problem sheets will be available on Blackboard and the solutions to the problem sheets will be uploaded after the problem classes.
If anybody wants to have some feedback on their un-assessed problem sheets, you can give me your solutions and I will have a look at it.

Course Rep: You should agree on a course rep in the first week.

Lecture notes: The lecture notes are available on Blackboard. They have gaps and we will fill these gaps during the lectures.

Textbooks:
All the material used in this module can be found in various textbooks.

Gillespie, A. (2013) *Business Economics (2nd Edition).* Oxford University Press.
Varian, H. R. (1992) *Microeconomic Analysis (3rd Edition)*. W. W. Norton & Co.
Varian, H. R. (2014) *Intermediate Microeconomics (9th Edition).* W. W. Norton & Co.

These books can be found (some electronically) in the college library. However, the course will be self-sufficient.

Assessment: 1 in-class test, worth 10%
1 two-hour final exam, worth 90%

Part 1 - Microeconomics

Supply and Demand – an introduction

Principal questions to be addressed by economics – what price should we be paying for goods, what price should a vendor be selling their goods for? Do the different motivations of the different parties give different answers to this question?

Supply refers to the quantity of a product that a vendor (or vendors) is willing and able to sell, at a given price in a given period of time.

Correspondingly, ***demand*** refers to the quantity of the product that the buyer (or society at large) is willing and able to purchase at a given price in a given period of time

- Willingness and ability both important
  e.g. pint of beer, flat in SK
- '…in a given period of time' also important, e.g. toilet paper in March 2020
- "… given location…"

The good's price is not the only determinant

- Demand can also depend on…
  - number and price of substitute goods, e.g. cider/beer, coffee/tea
  - number and price of complementary goods, e.g. car-petrol, toothbrush-toothpaste
  - level and distribution of income, e.g. flat in SK
  - consumer's tastes and habits e.g. chocolate, tobacco
  - consumer's expectations with regard to the future
- Alternative determinants for supply:
  - Changes to the overall cost of production
  - Change to the production technology, e.g. use robots to produce cars
  - Change in prices of input goods, e.g. steel
  - change in organisation
- Both supply and demand may also change over time

Law of Demand: *Ceteris paribus* (everything else being equal), an increase in price will usually lead to a drop in demand.

- This is often linked to either the **income effect** or the **substitution effect**:
  - A rise in price results in a decrease in the consumer's purchasing power: their income no longer covers the same quantity of the good in question. (income effect)
  - A rise in a good's price may result in consumers substituting it for a similar, less expensive good.
    (substitution effect)

Law of Supply: *Ceteris paribus*, an increase in price will lead to an increase in supply.

- This is because higher price (per unit) will incentivise greater production. It may be, e.g., that a manufacturer produces more than one product, and in order to optimise their profit, they have to split resources according to the revenue earned by each product. If this revenue split changes, so must the resource split, and therefore the amounts being produced.

In a market economy, the price of a good is determined according to its supply and demand, through the price mechanism:

- If supply exceeds demand, price drops as the producers compete to sell the good. This acts to encourage demand via the Law of Demand.

- If demand exceeds supply, the price increases as consumers compete with each other to obtain the good. This acts to incentivise production, as per the Law of Supply.

This interaction of the price mechanism with the Laws of Supply and Demand means that changes in both supply and demand will both cause and be caused by changes in the price of the good. This relationship determines the ***equilibrium price*** of the product; where supply and demand are equal.

Supply and demand curves

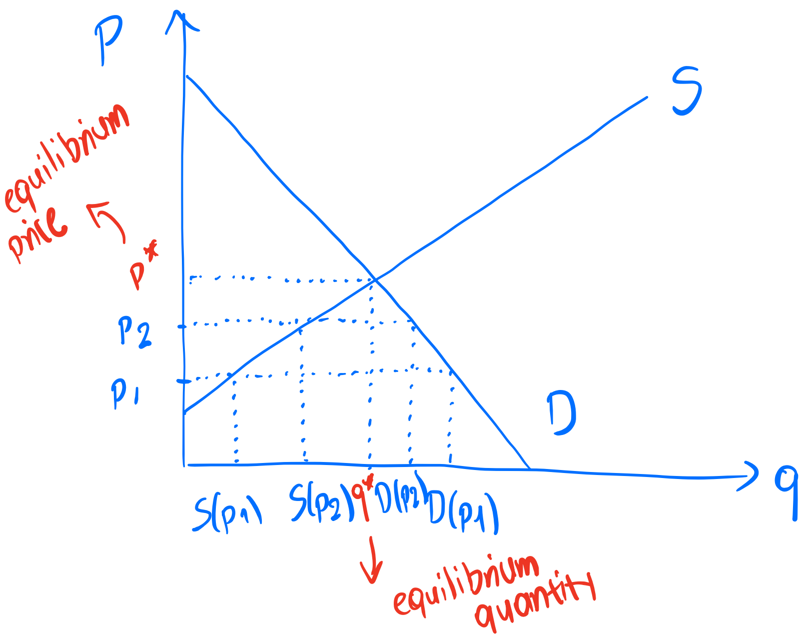
Examples of negative prices: oil, wasted rubbish

In mathematical terms, the demand and supply can be considered as functions D and S mapping a price $\boxed{p \in [0, \infty)}$ to some level of demand $D(p)$ or supply $S(p)$. It depends on the good of interest if these functions are integer-valued (discrete) or real-valued (continuous). If they can be inverted, their inverses are referred to as *inverse demand* and *inverse supply*. As such, they map from $\mathbb{N}$ or $\mathbb{R}$ to $[0, \infty)$.
Note that we ignore the fact that prices are also reported in discrete units and treat the price variable as a continuous quantity.

It is often convenient and revealing to analyse the supply and demand function graphically. For historic reasons (due to the economist Alfred Marshall) we use the convention that prices are depicted on the vertical axis and quantities on the horizontal axis. For trained mathematicians, this praxis is rather counter-intuitive. However, since the convention to do so is pervasive in the economic literature, we shall stick to it in this course.
The graphs of the supply and demand functions are referred to as supply and demand *curves.*

If one is to analyse stylised facts rather than precise quantitative results, one commonly uses linear functions for supply and demand for the sake of simplicity.

Examples of supply and demand curves:



Notes:

$D(p_1) > S(p_1)$

$D(p_2) > S(p_2)$

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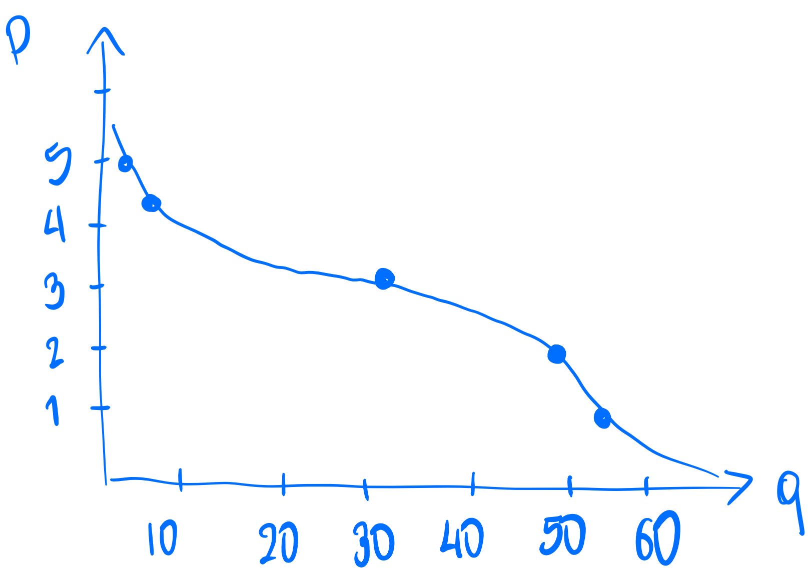
p (£)	0	1	2	3	4	5
q	61	53	50	32	9	2



Analysing the supply and demand curves:

It is of interest to economists to characterise the sensitivity of a product's supply or demand to shifts in its price; the measure of such sensitivity is known as the ***price elasticity of supply/demand.***

In general, the elasticity of a quantity refers to the **relative** magnitude of its reaction to a change in any variables on which it depends.

Consider the demand for a product, which depends on its price:

- if the demand for a product is fairly resilient and robust to price changes, then it is inelastic: demand changes relatively little for a given change in price
- Conversely, if the demand is particularly sensitive to changes in the price, then it is said to be elastic.

More rigorously, we denote and define the ***price elasticity of demand*** to be

$$\varepsilon_D = \varepsilon_D(p) = \frac{\text{proportional change in quantity demanded}}{\text{proportional change in price}}$$

$$= \frac{\frac{\partial D(p)}{\partial p} \Big/ D(p)}{1/p}$$

$$= \frac{\partial D(p)}{\partial p} \cdot \frac{p}{D(p)}$$

Law of demand $\Rightarrow \varepsilon_D < 0$

$$|\varepsilon_D(p)| = \begin{cases} >1 & \text{elastic demand} \\ <1 & \text{inelastic} \\ =1 & \text{unit elastic} \end{cases}$$

Determinants of ε_D:

- Number and closeness of substitute goods
- Proportion of income spent on the good
- Time period

We define the ***price elasticity of supply*** ε_S similarly; this measures the sensitivity of a good's supply function to changes in the good's price.

In addition, we can consider:

- the ***income elasticity of demand (supply)***, which measures the sensitivity of a good's demand (supply) function to changes in the consumer's income; and
- the ***cross-price elasticity of demand (supply)***, which measures the sensitivity of one good's demand (supply) function to changes in the price of another good.

$\varepsilon > 0 \Rightarrow$ sub. goods, $\varepsilon < 0 \Rightarrow$ comp. goods, $\varepsilon = 0 \Rightarrow$ independent goods

Demand, Price and Revenue

The revenue generated by a particular good is simply defined as the product of its price and quantity demanded

$$R(p) = Q(p) \cdot p = D(p) \cdot p$$

Since the demanded quantity of a good is inversely related to the good's price, an increase in price will not necessarily increase the revenue generated.

When will an increase in price result in an increase in revenue? How is this linked to the price elasticity?

$$\frac{\partial R(p)}{\partial p} = \frac{\partial D(p)}{\partial p} \cdot p + D(p) > 0 \iff$$

$$-\frac{\partial D(p)}{\partial p} \cdot p < D(p) \iff$$

$$-\frac{\partial D(p)}{\partial p} \cdot \frac{p}{D(p)} < 1$$

$$-\varepsilon_D(p) < 1$$

so an increase in price will lead to an increase in revenue iff demand is inelastic.

when demand for a good is elastic, revenue is increasing by decreasing the price.

Theory of the Firm

Production Functions, Cost, Revenue and Profit

The principle aim of a firm is to turn various inputs, such as raw materials and labour, into output that can then be sold, ideally for profit. Inputs to the production process are referred to as **factors of production**, and are broadly split into four categories:

Raw materials, labour, land, capital

The first three are self-explanatory; capital requires some explanation:

- Broadly speaking, capital refers to those inputs to production that may be consumed now, but that will deliver greater overall value to the firm if consumption is deferred.

- Capital goods are those inputs to production that are themselves produced goods, and which are durable, such as machinery, equipment.

- Capital finance refers to the financial assets of a firm that are themselves used to generate wealth; it differs from 'money' in general in that it is not used to purchase consumable goods and services, shares, bonds, patents.

We start by considering constraints that might be placed upon a firm's production capabilities; only certain combinations of input and output quantities may be technologically feasible, and so these are referred to as technological restraints, and the set of all inputs and outputs that satisfy such restraints are referred to as the ***production set***.

We will denote the vector of input factor quantities as $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and the vector of output quantities as $\underline{y} \in \mathbb{R}^m_{\geq 0}$; thus the **production set** is the collection of vectors $(\underline{x}, \underline{y}) \in \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m_{\geq 0}$ such that $\underline{y} \leq f(\underline{x})$, for some ***production function*** f. The production function, also known as the technology of the firm, prescribes the *maximum* level of output $\underline{y}$ for a given level of input $\underline{x}$.

For given $\underline{y} \in \mathbb{R}^m_{\geq 0}$, the set of all points $\underline{x} \in \mathbb{R}^n_{\geq 0}$, such that

$$f(x_1, x_2, \dots, x_n) = \underline{y}$$

is known as an ***isoquant.***

Note: we will mostly consider the single-output case ($m=1$), but the methodology extends to $m \geq 2$. Similarly, we will usually consider $\underline{x} \in \mathbb{R}^2$ or $\mathbb{R}^3$.

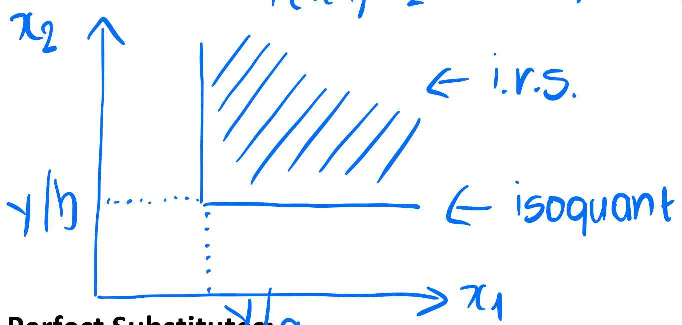
When $y \in \mathbb{R}_{\geq 0}$, the ***input requirement set*** is the set of all vectors $\underline{x}$ that produce at least y, that is $f^{-1}([y, \infty))$.

We now consider three examples of production functions that are often used in microeconomic analysis.

Leontief Technology (perfect complements)

Suppose we have two inputs x_1 and x_2: the Leontief production function takes the form

$$f(x_1, x_2) = \min(ax_1, bx_2), \quad a, b \geq 0$$



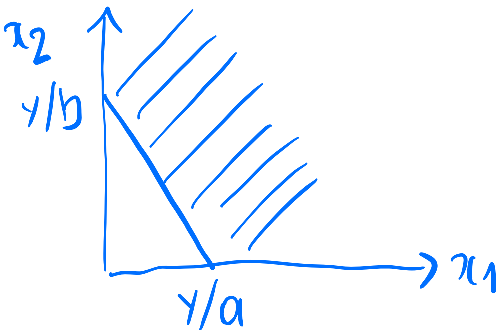
Example:

- output: bicycle (y=1)
- inputs: frames, wheels (a=1, b=1/2)
  (x_1) (x_2)

Perfect Substitutes: (Linear technology)

Suppose, in contrast, we have inputs to production that can be easily substituted for one another without affecting the level of output.

$$f(x_1, x_2) = ax_1 + bx_2, \quad a, b \geq 0$$



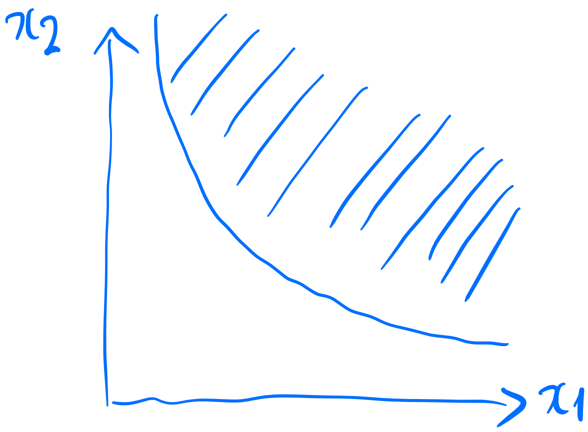
Example:

- output: coffee
- inputs: milk or soya milk

Cobb-Douglas Technology:

Suppose we have two inputs, x_1 and x_2: the Cobb-Douglas production function takes the form

$$f(x_1, x_2) = A x_1^a x_2^b, \quad A, a, b \geq 0$$



Properties of production functions / the input requirement sets:

- Monotonicity: If some of the inputs is increased, the maximum will not decrease.

$$\underline{x} \leq \underline{x}^* \Rightarrow f(\underline{x}) \leq f(\underline{x}^*)$$

- Convexity: If $(x_1, x_2) \in f^{-1}(\{y, \infty\})$ and $(x_1^*, x_2^*) \in f^{-1}(\{y, \infty\})$ then $(1-\lambda)(x_1, x_2) + \lambda(x_1^*, x_2^*) \in f^{-1}(\{y, \infty\})$.

This property is equivalent to the quasi-concavity of f.

Long-run and Short-run

Broadly speaking, in the economic literature, analysis is split into two scenarios, considering the behaviour of the firm or individual in either the short-run or the long-run. These are inexact periods of time, defined implicitly by the number of production inputs $x_1, \ldots, x_n$ that may vary within such a timeframe: in the long-run, all inputs may vary, whereas in the short run, at least one input will be held constant.

The Marginal Product

Question: how much can we increase output by varying the input factors?

Suppose we are operating with an element (x_1, x_2, y) in the production set of f and we wish to obtain a level of output $y' > y$ by increasing x_1...

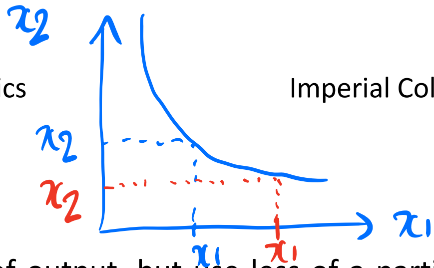
- The ***marginal product of factor i*** is defined as

$$MP_i(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_i}, \quad i = 1, 2$$

There is a slight issue with the marginal product – it is dependent on the units used to measure the input quantities and also on the units of the output quantities. In order to measure the effect of increasing each input independently of its units, we turn to the output elasticity wrt each input.

For a production function $f: \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$, the ***output elasticity*** with respect to input x_i is defined as the ratio of the relative change in output to the relative change in input:

$$\varepsilon_i(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_i} \cdot \frac{x_i}{f(x_1, x_2)}$$



Substitution

How about if we wish to keep the same level of output, but use less of a particular input factor?

We are interested in the rate of change of input x_2 with respect to x_1, in order to keep the output constant. This is the ***Marginal Rate of Technical Substitution (MRTS)***, or the technical rate of substitution.

Mathematically speaking, for a fixed $y \in \mathbb{R}$, we are interested in the derivative of a function g_y, where g_y is implicitly defined as

$$g_y(x_1) = x_2 \iff f(x_1, x_2) = y$$

The **implicit function theorem** asserts that such a g_y exists at least *locally*. Moreover, if f is a C^1-function, then also g_y is a C^1-function.

Then we obtain

$$f(x_1, g_y(x_1)) = y \quad \forall x_1$$

$$\partial_1 f(x_1, g_y(x_1)) + \partial_2 f(x_1, g_y(x_1))\, g_y'(x_1) = 0$$

If $\partial_2 f(x_1, g_y(x_1)) \neq 0$, then $g_y'(x_1) = -\dfrac{\partial_1 f(x_1, g_y(x_1))}{\partial_2 f(x_1, g_y(x_1))}$

We define the **MRTS** of a production function to be

$$MRTS(x_1, x_2) = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)}$$

Reasoning:

Start with the pair (x_1, x_2) and determine $y = f(x_1, x_2)$

Then, we proceed with the rationale from above, choosing g_y.

Consider the behaviour of the marginal product as a function of $x_1, \dots, x_n$: since the production function is nondecreasing, we have that the marginal product is nonnegative, and it is common to assume that it is nonincreasing. In other words, increasing an input x_1 from 100 to 101 is likely to result in a smaller increase in production than if we were increasing x_1 from 1 to 2. This is known as the ***law of diminishing marginal productivity***, and holds ceteris paribus.

(low-hanging fruits principle)

Similarly, it is common to assume that a firm's production has diminishing marginal rate of technical substitution: if we consider substituting factor x_2 for factor x_1 (i.e. decreasing x_1 and increasing x_2 such that the output is fixed), the larger the value for x_1 (before substitution), the smaller the absolute value of the MRTS ; in other words, the absolute value of the slope of the isoquant must decrease as x_1 increases.

Example: $f(x_1,x_2) = \sqrt{x_1 x_2}$

$$MRTS = - \frac{\frac{x_2}{\cancel{2\sqrt{x_1x_2}}}}{\frac{x_1}{\cancel{2\sqrt{x_1x_2}}}} = -\frac{x_2}{x_1}$$

Returns to Scale:

We have considered the effects on the production function of increasing individual factors whilst keeping others fixed, and we have considered the effect of substituting one factor for another whilst keeping the output level fixed. We now consider the effect on the production function of scaling all variable input factors by the same constant,

That means for some production function $f: \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\underline{x} \in \mathbb{R}^n_{\geq 0}$ you should consider the behaviour of the function

$$\mathbb{R}_{\geq 0} \ni t \longmapsto f(t\underline{x})$$

- The most common scenario is that of ***constant returns to scale***, and this is where an increase in all inputs results in a proportional increase in the output:

  $$f(t\underline{x}) = t f(\underline{x}) \quad \forall\ t>0 \quad \forall\ \underline{x} \in \mathbb{R}^n_{\geq 0}$$

  This is considered to be the most common scenario as, in most scenarios, the production process can simply be replicated: if we double all input factors (land, capital, labour and raw materials) then the production process can simply be duplicated.

- In some scenarios, ***increasing returns to scale*** may be observed; this is where

$$f(t\underline{x}) > tf(\underline{x}) \quad \forall\ t>1 \quad \forall\ \underline{x} \in \mathbb{R}^n_{\geq 0}$$

Example: Merging → saving admin costs

- ***Decreasing returns to scale*** refers to the case where

$$f(tx) < tf(\underline{x}) \quad \forall\ t>1 \quad \forall\ \underline{x} \in \mathbb{R}^n_{\geq 0}$$

HW Determine the returns to scale behavior of C-D production function: $f(x_1, x_2) = A x_1^a x_2^b$.

In order to fully characterise the scalability of a firm's production process, we wish to find a quantitative measure of the returns to scale; we turn again to the use of an elasticity measure.

For some $\underline{x} \in \mathbb{R}^n_{\geq 0}$, we consider $h_x(t) = f(t\underline{x})$.

If f is differentiable function, we can define its elasticity of scale at $\underline{x} \in \mathbb{R}^n_{\geq 0}$ as:

$$e(\underline{x}) = \frac{dh(t)}{dt} \cdot \frac{t}{h(t)} \Big|_{t=1} = \frac{h'(1)}{h(1)}$$

$$= \frac{df(t\underline{x})}{dt} \Big|_{t=1} \cdot \frac{1}{f(\underline{x})}$$

$$= \frac{\langle \nabla f(\underline{x}), \underline{x} \rangle}{f(\underline{x})}.$$

That means $e(\underline{x})$ is a local measure of the scale behaviour.

(i) $e(\underline{x}) \geq 1 \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0} \iff f(t\underline{x}) \geq t f(\underline{x}) \ \forall \ t \geq 1 \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0}$

(ii) $e(\underline{x}) \leq 1 \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0} \iff f(t\underline{x}) \leq t f(\underline{x}) \ \forall \ t \geq 1 \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0}$

(iii) $e(\underline{x}) = 1 \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0} \iff f(t\underline{x}) = t f(\underline{x}) \ \forall \ t > 0 \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0}$

Idea of the proof:
Consider for $\underline{x} \in \mathbb{R}^n$ the function $g_{\underline{x}} : (0, \infty) \to \mathbb{R}$ $g_{\underline{x}}(t) = \dfrac{f(t\underline{x})}{t f(\underline{x})}$

Additional potential properties of the production function

- Homogeneity: $f: \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is positively homogeneous of degree $k \in \mathbb{R}$ if
$$f(t\underline{x}) = t^k f(\underline{x}) \ \forall \ t > 0 \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0}$$
This has obvious links to the returns to scale, i.e.
if f is positively homogeneous function of degree $k \in \mathbb{R}$,
- it has IRTS if $k > 1$,
- it has DRTS if $k < 1$,
- it has CRTS if $k = 1$.

Homotheticity: $f: \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is homothetic if there is a positively homogeneous function $h: \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ and a strictly increasing function $g: \mathbb{R} \to \mathbb{R}$ such that
$$f(\underline{x}) = g(h(\underline{x})) \ \forall \ \underline{x} \in \mathbb{R}^n_{\geq 0}$$
Homogeneous and homothetic production functions are useful modelling scenarios as they prescribe isoquants that vary simply for differing levels of output.

Profit and profit maximisation

So far, we have only considered production functions and the influences of different factors to the production. However, what is the ultimate reason and motivation for a firm to produce products at all?
Most microeconomic analyses assume that the firm is profit driven – we will too!

The profit of a firm is simply its revenue minus its costs, where *all* costs of the firm are taken into account. It is often easy to overlook costs (e.g. labour costs for a self-employed person).

In application, the decision of how to maximise profit comes down to deciding how much output to produce and at what price, or how much input to buy and at what price. We therefore construct the profit function in terms of these variables:

$$\pi(\underline{x}, \underline{p}, \underline{w}) = \underline{p}\, f(\underline{x})^T - \underline{w}\, \underline{x}^T$$

$$\pi(\underline{x}, p, \underline{w}) = p\, f(\underline{x})^T - \underline{w}\, \underline{x}^T \text{ (single output case)}$$

We will assume the conditions of a purely competitive market, i.e. where all firms are assumed to be price-takers (their actions have a negligible effect on the prices).
In such a scenario, one need only determine the quantities of inputs/outputs in order to maximise profits.

Treating this as an unconstrained optimisation problem then, the first-order conditions for π to be maximised can be found straightforwardly:

$$\forall i = 1, \dots, n : \quad \frac{\partial \pi(\underline{x})}{\partial x_i} = 0 \iff \sum_{j=1}^{n} p_j \frac{\partial f_j(\underline{x})}{\partial x_i} = w_i$$

$$\text{or} \quad p \frac{\partial f(\underline{x})}{\partial x_i} = w_i \text{ (single output case)}$$

This first-order condition is often referred to as the fundamental condition of profit maximization:

- Profit is maximised if marginal revenue is equal to marginal cost
- Also stated as: 'the value of the marginal product wrt a factor is equal to its price'.

For fixed $(\underline{p}, \underline{w})$, this condition is easily solved to provide a necessary condition for

the profit-maximising input vector $\underline{x}^*$, which can then be used to establish the profit-maximising output.

So for the single-input case, the second-order condition for profit maximisation is that

$$f''(\underline{x}^*) \leq 0 \text{ (necessary) or } f''(\underline{x}^*) < 0 \text{ (sufficient)}$$

For $\underline{x} \in \mathbb{R}^n_{\geq 0}$, $n > 1$, the corresponding condition is that Hessian $\to M$ of $\underline{x} \mapsto p f(\underline{x})^T$ is negative semi-definite (necessary) or negative definite (sufficient).
i.e. $\underline{x}^T M \underline{x} \leq 0$ $\qquad$ $\underline{x}^T M \underline{x} < 0$

Quite often, this (local) concavity of the production function is part of our assumptions.

There are some **caveats** with this canonical procedure:

1) The production function might not be differentiable, e.g. Leontief production function.

2) The input goods might be discrete rather than continuous.

3) We might have boundary solutions (usually at 0 meaning that it is optimal not to use a certain factor in the production at all).
e.g. $f(x) = x$ and $w > p$.

4) A best strategy might not exist, e.g. $f(x) = x$ and $w < p$.

5) A solution might exist, but it is not unique, e.g. $f(x) = x$ and $w = p$.

For a specific production function $f: \mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ we introduce the map

$$\underline{x}^* : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to 2^{\mathbb{R}^n_{\geq 0}}$$

yielding the / an optimal specification of input quantities, given a price $\underline{p} \in \mathbb{R}^m_{\geq 0}$ and prices for input factors $\underline{w} \in \mathbb{R}^n_{\geq 0}$. That is

$$\underline{x}^*(\underline{p}, \underline{w}) = \underset{\underline{x} \in \mathbb{R}^n_{\geq 0}}{\operatorname{argmax}}\, \pi(\underline{x}, \underline{p}, \underline{w}) = \underset{\underline{x} \in \mathbb{R}^n_{\geq 0}}{\operatorname{argmax}}\, \underline{p} f(\underline{x})^T - \underline{w}\,\underline{x}^T$$

In the light of the caveats mentioned above, our standard *assumption* is that the optimal input specification exists and is unique. Under this assumption, the values of $\underline{x}^*$ are singletons and we can identify the singletons with their unique element.
Consequently, we can consider $\underline{x}^*$ as a function – the ***factor demand function*** – being a map $\underline{x}^* : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$.

Under this assumption we can also define the ***output supply*** as the function

$$y^*: \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0} \quad (\underline{p}, \underline{w}) \mapsto y^*(\underline{p}, \underline{w}) = f(\underline{x}^*(\underline{p}, \underline{w}))$$

Moreover, we can define the **profit function**

$$\pi^*: \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}$$

$$(\underline{p}, \underline{w}) \mapsto \pi^*(\underline{p}, \underline{w}) = \pi(\underline{x}^*(\underline{p}, \underline{w}), \underline{p}, \underline{w})$$

$$= \max_{\underline{x} \in \mathbb{R}^n_{\geq 0}} \pi(\underline{x}, \underline{p}, \underline{w})$$

$$= \max_{\underline{x} \in \mathbb{R}^n_{\geq 0}} \underline{p} f(\underline{x})^T - \underline{w}\,\underline{x}^T$$

Properties: (Proof 1-4, PS2)

1. The factor demand function is positively homogeneous of degree 0.
   $\underline{x}^*(t\underline{p}, t\underline{w}) = \underline{x}^*(\underline{p}, \underline{w}) \quad \forall\, \underline{p}, \underline{w} \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \quad \forall\, t > 0$
   Note: The optimal input bundle is invariant under change of currency.
2. The profit function is positively homogeneous of degree 1. $\pi^*(t\underline{p}, t\underline{w}) = t\,\pi^*(\underline{p}, \underline{w})$
3. The profit function is non-decreasing in $\underline{p}$ and non-increasing in $\underline{w}$.
4. The profit function is convex. in both arguments $\underline{p}$ and $\underline{w}$.
5. Under some regularity assumptions, the profit function is continuous.

Note: Under change of currency, the profit should change accordingly.

<u>Proof:</u> 5. Follows with the *Berge Maximum Theorem*. The other assertions are exercises.

Hotelling's Lemma (a special case of Envelope Theorem)

The output supply and factor demand functions can be obtained directly from the maximised profit function through partial differentiation with respect to the price vector.

$$\pi^*(\underline{p},\underline{w}) = \underline{p}\, f(\underline{x}^*(\underline{p},\underline{w}))^T - \underline{w}\,\underline{x}^*(\underline{p},\underline{w})^T$$

$$\frac{\partial \pi^*(\underline{p},\underline{w})}{\partial p_j} =$$

$$f_j(\underline{x}^*(\underline{p},\underline{w})) + \sum_{k=1}^{m} p_k \sum_{i=1}^{n} \partial_i f_k(\underline{x}^*(\underline{p},\underline{w})) \cdot \frac{\partial x_i^*(\underline{p},\underline{w})}{\partial p_j} - \sum_{i=1}^{n} w_i \frac{\partial x_i^*(\underline{p},\underline{w})}{\partial p_j} =$$

$$f_j(\underline{x}^*(\underline{p},\underline{w})) + \sum_{i=1}^{n} \Big[\overbrace{\sum_{k=1}^{m} p_k \partial_i f_k(\underline{x}^*(\underline{p},\underline{w}))}^{=w_i} - w_i \Big] \cdot \frac{\partial x_i^*(\underline{p},\underline{w})}{\partial p_j} =$$

$$f_j(\underline{x}^*(\underline{p},\underline{w}))$$

$$\frac{\partial \pi^*(\underline{p},\underline{w})}{\partial w_j} = -x_j^*(\underline{p},\underline{w}) + \sum_{i=1}^{n} \Big[\sum_{k=1}^{m} p_k \partial_i f_k(\underline{x}^*(\underline{p},\underline{w})) - w_i \Big] \cdot \frac{\partial x_i^*(\underline{p},\underline{w})}{\partial w_j}$$

$$= -x_j^*(\underline{p},\underline{w})$$

Note: (Property 2: profit function pos. hom of degree 1 $\Rightarrow$) output supply and input demand pos. hom. of degree 0
(p.18)

One convenient corollary of Hotelling's Lemma is that each row of the maximised profit function Hessian (wrt $\underline{p}$) is equivalent to the gradient of the firm's supply function for the corresponding output. This makes it easier to derive some properties of profit-maximizing firms, such as…

The Le Chatelier's principle:

The long-run supply response to a change in price is at least as large as the short-run supply response.

$$\frac{\partial y_i^*(\underline{p},\underline{w})}{\partial p_i} \geq 0$$ (Proof: Use Hotelling's lemma and convexity of profit function)

Weak Axiom of Profit Maximization (WAPM):

Q: Can we make inference about production function?

So far, we have started with a given production function f and we have derived some results about the factor demand function or the output supply function. Now, we turn the perspective and assume that we can observe a firm's 'behaviour'. The **Weak Axiom of Profit Maximization (WAPM)** is a necessary condition for the rational, i.e. profit maximizing behaviour of that company.
In a first step, we can use the WAPM to check if a company is profit maximizing by checking if the observed dataset satisfies the WAPM.
In a second step, we can even consider some attempts of statistical inference for the production function using a dataset that satisfies WAPM.

Suppose we observe the *net output* vectors $\underline{z}^t = \left(\underline{y}^t, -\underline{x}^t\right) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$ and their corresponding price vectors $\underline{r}^t = \left(\underline{p}^t, \underline{w}^t\right) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$ for some firm at discrete time points $t = 1, \dots, T$. Assuming that the firm is acting to maximise profits, we can deduce that

profit at time point t $\leftarrow$

$$\underline{r}^t\left(\underline{z}^t\right)^\top \geq \underline{r}^t\left(\underline{z}^s\right)^\top \quad \forall s, t = 1, \dots, T.$$

This is the Weak Axiom of Profit Maximization:

$$\underline{r}^t(\underline{z}^t - \underline{z}^s) \geq 0 \quad \text{and} \quad -r^s(\underline{z}^t - \underline{z}^s) \geq 0 \quad \forall s,t = 1,\dots,T$$

Writing WAPM with indices switched yields

$$\left(\underline{z}^t - \underline{z}^s\right)\ \left(\underline{r}^t - \underline{r}^s\right)^\top \geq 0$$

Note: If the price of an output good increases, then the supply of this good should not decrease.
If the price of an input good increases, then the ~~supply~~ demand of this good should not increase.
(given that no other prices change).

No assumptions made about the production function.

Cost Minimisation

We previously looked at a direct, unconstrained approach to profit maximisation: given fixed costs for our inputs $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and a fixed price for our output $y \in \mathbb{R}$, what production choices lead to maximum profit? In non-competitive markets, however, since the output price $p \in \mathbb{R}$ is not necessarily fixed, it is useful to split this into two constituent problems:

- For fixed $\underline{w} \in \mathbb{R}^n_{\geq 0}$, what is the minimum total cost to the firm of producing a level of output y?
- Given this knowledge, what is the most profitable level of output?

We consider the first part of this problem here.
Note that, for $n = 1$, this is trivial; we therefore assume here that $n \geq 2$.

For input vector $\underline{x} \in \mathbb{R}^n_{\geq 0}$ with associated prices $\underline{w} \in \mathbb{R}^n_{\geq 0}$, we are interested in solving the optimisation problem:

constraint: $f(\underline{x}) = y \iff f(\underline{x}) - y = 0$

$$\text{Find } \underset{\underline{x} \in f^{-1}(\{y\})}{\operatorname{argmin}} \ \underline{w}\,\underline{x}^T$$

To solve this constrained optimisation problem, we convert it into a Lagrangian problem; we incorporate the constraint into the objective function, and subsequently treat it as an unconstrained minimisation.

First, define the Lagrangian, $\mathcal{L}$:

Langrangian multiplier

$$\mathcal{L} : \mathbb{R}^n_{\geq 0} \times \mathbb{R} \to \mathbb{R} \qquad \mathcal{L}(\underline{x}, \lambda) = \underline{w}\,\underline{x}^T - \lambda\left(f(\underline{x}) - y\right)$$

Then, find the first-order conditions for minimising $\mathcal{L}$:

$$\frac{\partial \mathcal{L}(\underline{x},\lambda)}{\partial x_i} = w_i - \lambda \partial_i f(\underline{x}) = 0 \quad \forall\, i = 1, \dots, n$$

$$\frac{\partial \mathcal{L}(\underline{x},\lambda)}{\partial \lambda} = f(\underline{x}) - y = 0 \quad \text{(constraint)}$$

Solve these n +1 equations for the n+1 unknowns

$$\lambda \partial_i f(\underline{x}) = w_i \quad \forall\, i = 1, \dots, n$$

$$f(\underline{x}) = y$$

The conditions on x_i for cost-minimisation look reassuringly similar to those obtained for competitive profit maximisation; they are not the same however. Here, λ is simply a dummy variable, which we must get rid of if we are to make any progress; we cannot solve explicitly for x_i in terms of known prices.

If f has a known, differentiable form…

- Use solved condition to find $\underline{x}$ in terms of λ and $\underline{w}$.
- Substitute into the constraint and rearrange to find λ in terms of y and $\underline{w}$.
- Re-substitute into the solved condition to find $\underline{x}$ in terms of y and $\underline{w}$.

Thus, we can determine the level of each input factor in terms of the input factor prices and the desired level of output; these relationships constitute the ***conditional factor demand function***, which we denote $x^*(\underline{w}, y)$. Formally, this is a function

$$x^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \quad (\underline{w}, y) \mapsto \underset{\underline{x}\in f^{-1}(\{y\})}{\operatorname{argmin}}\ \underline{w}\ \underline{x}^\top$$

$x^*(p, \underline{w})$ factor demand function (profit max. problem) vs $x^*(\underline{w}, y)$ conditional factor demand function (cost min. problem)

The minimum total cost to the firm of producing a level of output y with input prices $\underline{w}$ can subsequently be obtained as the price-weighted conditional factor demand functions:

$$c^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \quad c^*(\underline{w}, y) = \min_{\underline{x}\in f^{-1}(\{y\})} \underline{w}\ \underline{x}^\top = \underline{w}\ \underline{x}^*(\underline{w}, y)^\top$$

Example: Cobb-Douglas $f(x_1,x_2) = A x_1^a x_2^b$, $A, a, b > 0$

Find the conditional factor demand function.

Solution: Find minimiser of $\underline{w}\,\underline{x}^T$ s.t. $f(\underline{x}) = y$.

FOC: $\lambda A a x_1^{a-1} x_2^b = w_1$ (1) Constraint: $A x_1^a x_2^b = y$ (3)

$\lambda A b x_1^a x_2^{b-1} = w_2$ (2)

(1)÷(2) $\Rightarrow \frac{w_1}{w_2} = \frac{a x_2}{b x_1} \Rightarrow \boxed{x_2 = x_1 \frac{b}{a}\frac{w_1}{w_2}}$ (4)

Sub. (4) into (3): $A x_1^a \left(x_1 \frac{b}{a}\frac{w_1}{w_2}\right)^b = y \Rightarrow A x_1^{a+b}\left(\frac{b}{a}\frac{w_1}{w_2}\right)^b = y \Rightarrow$

$x_1^*(w_1,w_2,y) = y^{1/a+b} A^{-1/a+b}\left(\frac{b}{a}\frac{w_1}{w_2}\right)^{-b/a+b}$, similarly $x_2^*(w_1,w_2,y) = y^{1/a+b} A^{-1/a+b}\left(\frac{b}{a}\frac{w_1}{w_2}\right)^{1-b/a+b}$

We note that the first-order conditions above can be restated in terms of the marginal rate of technical substitution:

$$MRTS(x_1^*, x_2^*) = -\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = -\frac{\partial_1 f(x_1,x_2)}{\partial_2 f(x_1,x_2)} = -\frac{w_1}{w_2}$$

That means the marginal rate of technical substitution coincides with the **economic rate of substitution**.

Possible problems when finding the conditional factor demand function:

1) The production function might not be differentiable, e.g. Leontief production function.

2) We might have a boundary solution (meaning that some input quantity is 0 in the optimum).

3) If the production function is continuous, surjective on $[0,\infty)$, and $w \gg 0$, there is always a cost minimising strategy (in contrast to the profit maximisation problem).

Proof: The objective function $\underline{x} \mapsto \underline{w}\,\underline{x}^T$ is continuous.
A continuous function attains a minimum and a maximum over a compact set in $\mathbb{R}^n$ → closed and bounded set
Since f is continuous, the pre-image $f^{-1}(\{y\})$ is closed.
We can pick some arbitrary $\underline{x}' \in f^{-1}(\{y\})$ and restrict attention to the set:
$$f^{-1}(\{y\}) \cap \{\underline{x} \in \mathbb{R}^n_{\geq 0} \mid \underline{w}\,\underline{x}^T \leq \underline{w}\,\underline{x}'^T\}.$$
This set is compact.

4) The optimal strategy might not be unique.

<u>Properties of the cost function</u>

The cost function $c^*(\underline{w}, y)$ is…

- Nondecreasing in $\underline{w}$:

$\forall\ \underline{w}, \underline{w}' \in \mathbb{R}^n_{\geq 0}\ \forall\ y \geq 0:\ \underline{w}' \geq \underline{w} \Rightarrow c^*(\underline{w}', y) \geq c^*(\underline{w}, y)$

- Homogeneous of degree 1 in $\underline{w}$

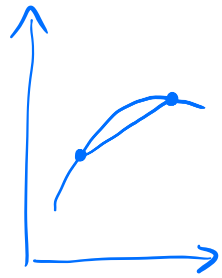
$\forall\ \underline{w} \in \mathbb{R}^n_{\geq 0}\ \forall\ y \geq 0\ \forall\ t > 0:\ c^*(t\underline{w}, y) = t c^*(\underline{w}, y) \to$ Under change of currency, the cost should change in the same manner.

- Concave in $\underline{w}$

$\forall\ \underline{w}, \underline{w}' \in \mathbb{R}^n_{\geq 0}, \forall\ y \geq 0\ \forall\ t \in [0,1]\ c^*(t\underline{w} + (1-t)\underline{w}', y) \geq t c^*(\underline{w}, y) + (1-t) c^*(\underline{w}', y)$

- Continuous in $\underline{w}$

$\forall\ \underline{w} \in \mathbb{R}^n_{>0}$ if $\underline{w}_0 \in \mathbb{R}^n_{>0}$ then $\lim_{\underline{w} \to \underline{w}_0} c^*(\underline{w}, y) = c^*(\underline{w}_0, y)$



- **Proof:** Exercise.

<u>Shephard's Lemma</u> (application of the Envelope theorem)

We can obtain the conditional factor demand functions from the cost function through differentiation with respect to the price vector $\underline{w}$; this is Shephard's Lemma: if $c^*(\underline{w}, y)$ is differentiable in $\underline{w}$ at $(\underline{w}, y)$ and $w_i > 0\ \forall\ i \in \{1, \dots, n\}$, then

$$x_i^*(\underline{w}, y) = \frac{\partial c^*(\underline{w}, y)}{\partial w_i}$$

"Alternative" proof:

Let $\underline{x}'$ be the cost-minimising input bundle at prices $\underline{w}'$ and output y. Define:

$$g(\underline{w}) = c^*(\underline{w}, y) - \underline{w}\,\underline{x}'^T$$

Note that $g(\underline{w}) \leq 0$ and $g(\underline{w}') = 0$.

FOC yield for the maximum of the f:

$$\left.\frac{\partial g(\underline{w})}{\partial w_i}\right|_{\underline{w} = \underline{w}'} = 0 \quad (\text{or } \partial_i g(\underline{w}') = 0)$$

That means: $\dfrac{\partial c^*(\underline{w}, y)}{\partial w_i} = x_i'$ at $\underline{w} = \underline{w}'$.

<u>Remark</u>: One can use this proof to establish the concavity (Property #3).

Weak Axiom of Cost Minimisation (WACM)

We can make very similar considerations as in the case of the Weak Axiom of Profit Maximisation. That means the ***Weak Axiom of Cost Minimisation (WACM)*** gives a necessary condition on data to stem from a cost minimising (and thus rationally operating) firm.

Assume we have observations of prices $\underline{w}^t \in \mathbb{R}^n_{\geq 0}$ and inputs $\underline{x}^t \in \mathbb{R}^n_{\geq 0}$ and outputs y^t at time points $t = 1, \dots, T$. Then the WACM states that

$$\underline{w}^t (\underline{x}^t)^T \leq \underline{w}^t (\underline{x}^s)^T \quad \forall\, s,t = 1,\dots,T \text{ s.t. } \boxed{y^s \geq y^t}$$

This implies that (proof hw)

$$(\underline{w}^t - \underline{w}^s)(\underline{x}^t - \underline{x}^s) \leq 0 \quad \forall\, s,t = 1,\dots,T \text{ s.t. } \boxed{y^s = y^t}$$

Note: If all prices but w_i are held constant, then in order to minimize costs whilst keeping output constant, the change in x_i must be in the opposite direction to the change in w_i.

e.g. If the input price of a good increases, then we should use less of that good.

<u>No</u> assumptions about the production function.

WAPM ⇒ WACM

<u>Proof:</u> WAPM: $p^t y^t - w_1^t x_1^t - w_2^t x_2^t \geq p^t y^s - w_1^t x_1^s - w_2^t x_2^s$

Consider a pair s,t s.t. $y^s \geq y^t$.

$$-w_1^t x_1^t - w_2^t x_2^t \geq p^t y^s - p^t y^t - w_1^t x_1^s - w_2^t x_2^s$$

$$-w_1^t x_1^t - w_2^t x_2^t \geq p^t \underbrace{(y^s - y^t)}_{\geq 0} - w_1^t x_1^s - w_2^t x_2^s$$

$$-w_1^t x_1^t - w_2^t x_2^t \geq -w_1^t x_1^s - w_2^t x_2^s$$

$$w_1^t x_1^t + w_2^t x_2^t \leq w_1^t x_1^s + w_2^t x_2^s$$

Long-run vs. short-run costs

We have considered cost minimisation under the assumption that all of our inputs to production are allowed to vary in quantity – recall that this is a long-run scenario. In the short run, at least one factor will remain fixed.
Let $F, V \subseteq \{1, \ldots, n\}$ be index sets with $F \cup V = \{1, \ldots, n\}$ and $F \cap V = \emptyset$. Here, the set V consists of the indices of variable short-run factors and F comprises the indices of the fixed long-run factors.
Then, for $\underline{x} \in \mathbb{R}^n_{\geq 0}$ we shall write $\underline{x} = (\underline{x}_F, \underline{x}_V) \in \mathbb{R}^F_{\geq 0} \times \mathbb{R}^V_{\geq 0}$.

The level of the fixed factor will influence both the minimised cost, given by the ***short-run cost function***

$$c_s^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^F_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$$

$$c_s^*(\underline{w}, \underline{x}_F, y) = \min_{\underline{x}_V \in \mathbb{R}^n_{\geq 0} \text{ s.t. } f(\underline{x}_F, \underline{x}_V) = y} \underline{w}\,\underline{x}^T = \underline{w}_F \underline{x}_F^T + \min_{\underline{x}_V \in \mathbb{R}^n_{\geq 0} \text{ s.t. } f(\underline{x}_F, \underline{x}_V) = y} \underline{w}_V \underline{x}_V^T$$

…and the cost-minimising choices of the variable factors, given by the ***short-run conditional factor demand functions***

$$x_s^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^F_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^V_{\geq 0}$$

$$x_s^*(\underline{w}, \underline{x}_F, y) = \operatorname*{argmin}_{\underline{x}_V \in \mathbb{R}^V_{\geq 0} \text{ s.t. } f(\underline{x}_F, \underline{x}_V) = y} \underline{w}\,\underline{x}^T = \operatorname*{argmin}_{\underline{x}_V \in \mathbb{R}^V_{\geq 0} \text{ s.t. } f(\underline{x}_F, \underline{x}_V) = y} \underline{w}_V \underline{x}_V^T$$

Note that

$$c^*(\underline{w}, y) \leq c_s^*(\underline{w}, \underline{x}_F, y)$$

$$x^*(\underline{w}, y) \quad ? \quad x_s^*(\underline{w}, \underline{x}_F, y)$$

we cannot establish a similar relation between the short-run conditional factor demand function and its long-run version.

Average and marginal costs

The cost function is used to gain insight into the economic capabilities of the firm; indeed, much of the firm's economic behaviour can be gleaned from $c^*(\underline{w}, y)$. It is particularly important to be able to analyse the behaviour of the cost function as the level of y changes, and so we now define a series of derived quantities for both short-run and long-run analyses.

Assuming the costs $\underline{w} = (\underline{w}_F, \underline{w}_V)$ to be fixed, we shall supress the costs in the notation and we define the ***short-run average cost*** $SAC(y)$ to be the per-unit cost of producing y units of output:

$$SAC(y) = \frac{c_S^*(\underline{w}, \underline{x}_F, y)}{y}$$

Assuming that the firm is cost-minimising, $\underline{x}_V = \underline{x}_S^*(\underline{w}, \underline{x}_F, y)$.

$$SAC(y) = \frac{\underline{w}_F \underline{x}_F^T}{y} + \frac{\underline{w}_V \underline{x}_S^*(\underline{w}, \underline{x}_F, y)}{y} = SAFC(y) + SAVC(y)$$

short-run average fixed costs

short-run average variable costs

It is also useful to consider the *rate* at which a firm's costs increase (or decrease) with respect to its output; the ***short-run marginal cost*** $SMC(y)$ is defined as

$$SMC(y) = \frac{\partial c_S^*(\underline{w}, \underline{x}_F, y)}{\partial y}$$ (assuming that the cost function is differentiable)

In the long-run, we have only variable input factors, i.e. $V = \{1, \dots, n\}$, $F = \emptyset$, leading to $\underline{x} = \underline{x}_V$, therefore the long-run average and marginal costs are defined accordingly:

$$LAC(y) = \frac{c^*(\underline{w}, y)}{y} \quad \text{or } AC(y)$$

$$LMC(y) = \frac{\partial c^*(\underline{w}, y)}{\partial y} \quad \text{or } MC(y)$$

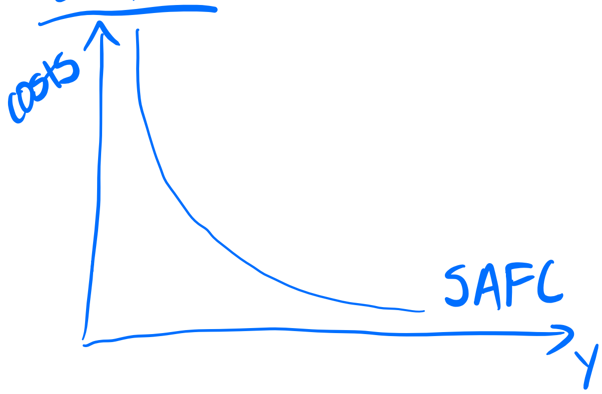
Geometry of costs

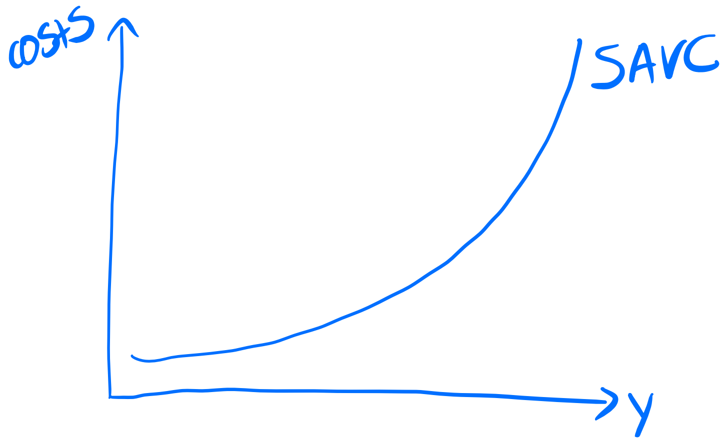
The shape of the average and marginal cost curves can be illuminating, and indicative of the economic capabilities of a firm with a given production function. Suppose we are operating in the short run.

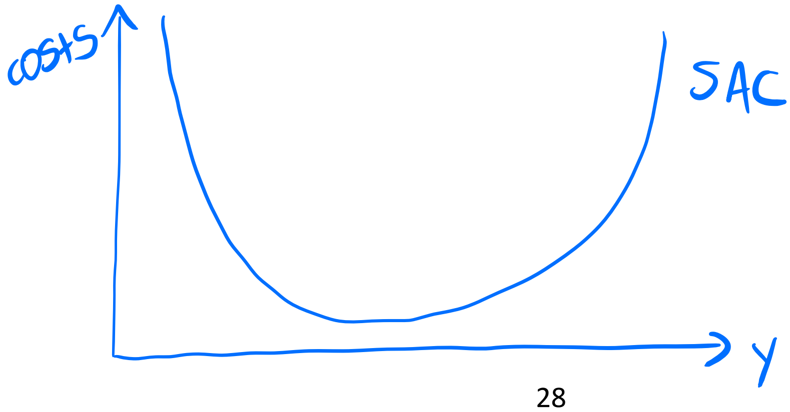
As y increases, SAFC will clearly decrease – what will happen to the variable costs?

Law of diminishing marginal productivity states that as variable inputs increase, the marginal productivity of each subsequent unit of input will decrease.
This is because variable inputs are combined with a given quantity of fixed inputs.
This implies that as we increase output, the input required to maintain a unit increase in output will itself increase. Thus, the per-unit variable costs will increase as y increases.

Sketch:







SAC = SAFC + SAVC has a minimum

Consider the minimum of the SAC - where does this occur?

$$FOC: \frac{\partial}{\partial y} SAC(y) = \frac{\partial}{\partial y}\left(\frac{C_S^*(\underline{w}, \underline{x}_F, y)}{y}\right) = \frac{\frac{\partial}{\partial y} C_S^*(\underline{w}, \underline{x}_F, y)\, y}{y^2} - \frac{C_S^*(\underline{w}, \underline{x}_F, y)}{y^2}$$

$$\frac{\partial}{\partial y} SAC(y) = 0 \overset{y>0}{\iff} \frac{\partial}{\partial y} C_S^*(\underline{w}, \underline{x}_F, y) = \frac{C_S^*(\underline{w}, \underline{x}_F, y)}{y}$$

…so we have that, at its local minimum, the average cost curve is intersected by the marginal cost curve. Similar analysis reveals that:

- $SMC(y) < SAC(y) \Leftrightarrow$ short-run average costs are decreasing in y
- $SMC(y) > SAC(y) \Leftrightarrow$ short-run average costs are increasing in y
- $SMC(y) = SAC(y) \iff$ short-run average costs have a minimum

The link between the average and marginal costs in the short-run can be further probed; consider how each behave at $y = 0$? or as $y \downarrow 0$?

$$SAC(y) = \frac{\underline{w}_F^T \underline{x}_F}{y} + \frac{\underline{w}_V\, \underline{x}_S^*(\underline{w}, \underline{x}_F, y)}{y}$$

$$\left(\text{as } y \to 0 \qquad \downarrow \infty \qquad \downarrow ?\right)$$

L'Hospital's rule implies for $y \to 0$:

$$\lim_{y \to 0} \frac{\underline{w}_V\, \underline{x}_S^*(\underline{w}, \underline{x}_F, y)}{y} = \lim_{y \to 0} \frac{\underline{w}_V \frac{\partial}{\partial y} \underline{x}_S^*(\underline{w}, \underline{x}_F, y)}{1}$$

So, as $y \to 0$,

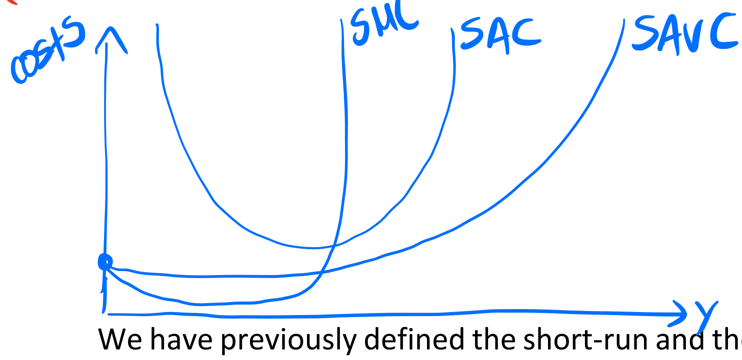
- the short-run average costs explode in the presence of fixed costs…
- but the short-run average variable costs and short-run marginal costs are equal

It shouldn't be surprising that the link between average and marginal costs is chiefly through the variable costs…the fixed costs do not contribute to the marginal costs! Indeed, we can further note:

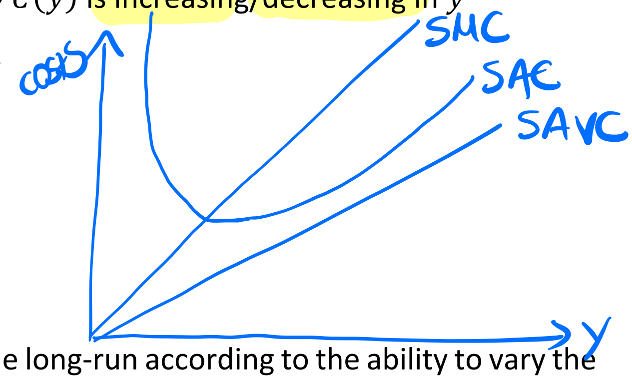
- The area under the marginal cost curve (MC) will give the total variable costs

-
- As we saw above with the short-run average and marginal costs,
    - $SMC(y) = SAVC(y)$ at the local minimum of $SAVC(y)$
    - $SMC(y) \gtrless SAVC(y) \Leftrightarrow SAVC(y)$ is increasing/decreasing in y

(Default)



(Alternative)



We have previously defined the short-run and the long-run according to the ability to vary the factors of production. We have considered the long-run to be the period of time in which all factors can be varied. We revisit the notions of fixed and variable factors, and consider their role in short-run and long-run cost analyses.

Fixed costs are those costs that do not scale with the firm's output. One cannot influence the level of production through altering a fixed factor of production. Even if a firm was to drop all output to $y = 0$, fixed costs would still require payment.

In contrast, variable costs are dependent on the firm's level of output, as the output is influenced by changing the variable factors of production.

In the short-run, there are both fixed and variable factors of production, and thus also fixed and variable costs. In the long-run, some of the fixed factors may be more easily varied, and so can be used to influence the level of output. Thus, factors that are fixed in the short-run are often variable in the long-run.

There may well be, however, some costs that are constant with respect to the level of output even in the long-run, as long as the firm is producing a positive level of output (i.e. is still in business); these are referred to as ***quasi-fixed costs***, and they correspond to quasi-fixed factors of production.

Summary:

Short-run: Variable costs + Fixed costs

Long-run: Variable costs + Quasi-fixed costs, e.g. licence, rent

Consider, now, the long-run $AC(y)$ and $MC(y)$ curves. The existence of quasi-fixed costs implies that the average and marginal cost curves will have a similar shape in the long run as in the short-run.

Recall from before:

$$c^*(\underline{w}, y) \leq c_s^*(\underline{w}, \underline{x}_F, y)$$

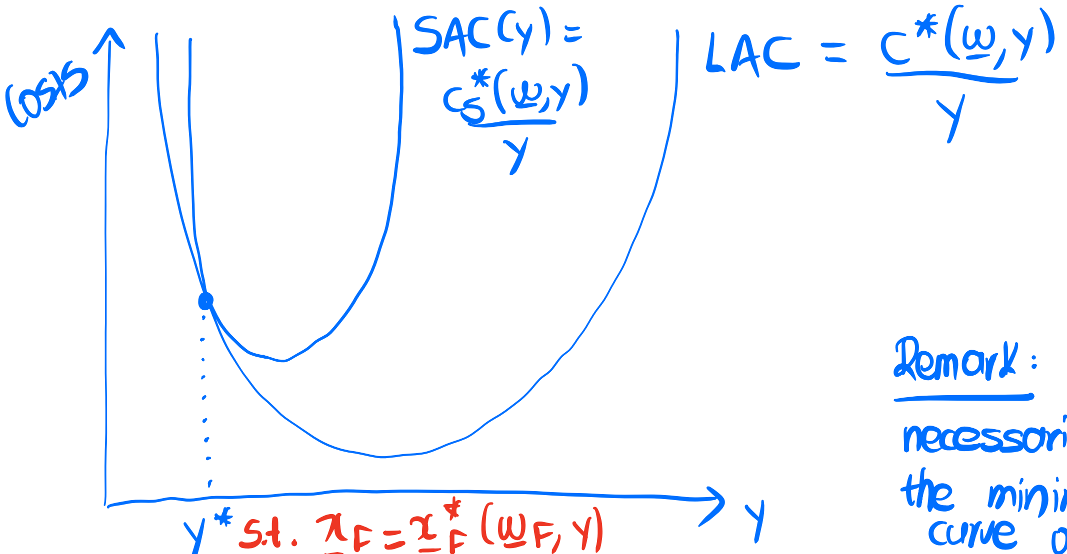
i.e. short-run costs are always greater than or equal to the long-run costs. This still holds, even in the presence of quasi-fixed costs; factors that are 'fixed' in the long-run will also be fixed in the short-run.

Recall also, however, that $c^*(\underline{w},y) = c_S^*(\underline{w}, \underline{x}_F^*(\underline{w},y), y) \quad \forall\, y>0$

i.e. for each level of output there will be an optimal level of the fixed factor given by its conditional factor demand.

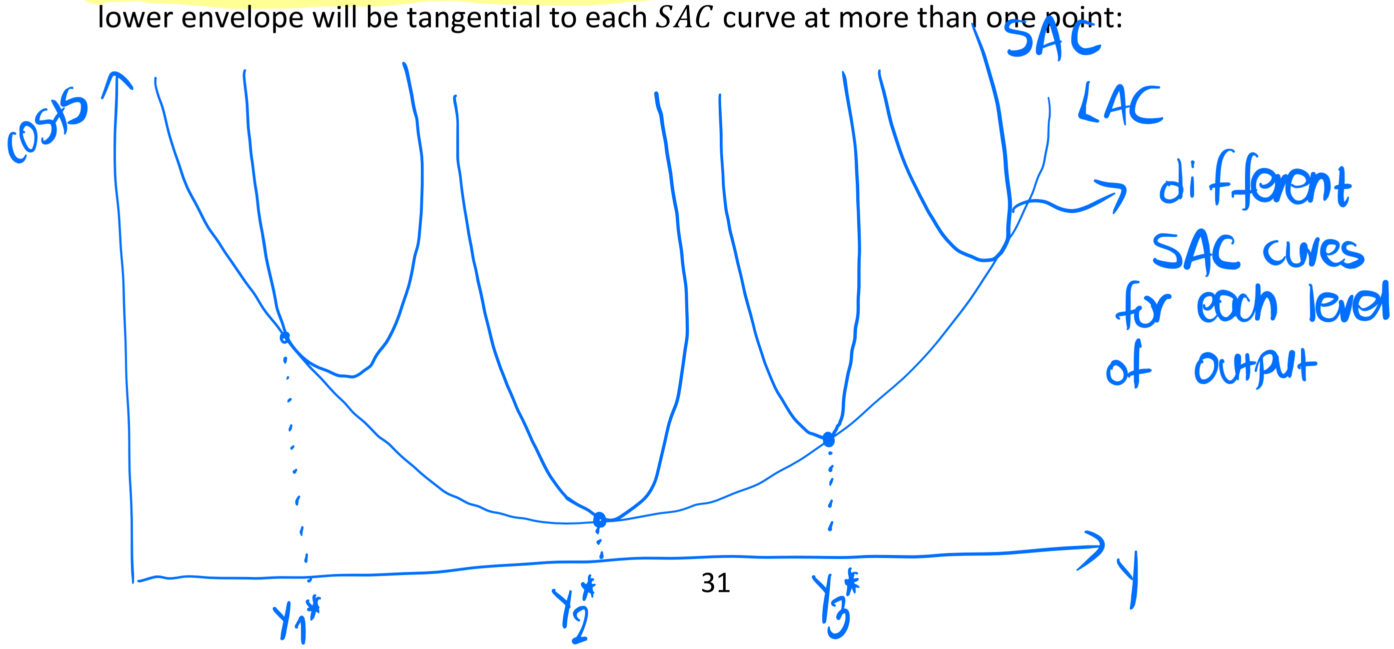
We conclude that the long-run $AC(y)$ curve:

- has a similar shape to $SAC(y)$;
- lies below or on the $SAC(y)$ curve at all points $y > 0$;
- and is tangential to the $SAC(y)$ at the point y^* for which $\underline{x}_F = \underline{x}_F^*(\underline{w}_F, y^*)$.



Remark: y^* doesn't necessarily have to be the minimum of the SAC curve or LAC curve

Note that, as $\underline{x}_F$ varies, the point y^* at which $SAC(y) = LAC(y)$ will move. If the fixed factors can vary continuously, then the SAC curve will trace out the LAC curve; we say that the LAC curve is the lower envelope of the SAC curves. If $\underline{x}_F$ can only be varied discretely, then this lower envelope will be tangential to each SAC curve at more than one point:



In order to characterise the behaviour of the long-run marginal costs, we note that the relationships that held between average and marginal costs in the short run will also hold in the long run:

- $LMC(y) = LAC(y) \Leftrightarrow$ long-run average costs are at a (local) minimum
- $LMC(y) < LAC(y) \Leftrightarrow$ long-run average costs are decreasing in y
- $LMC(y) > LAC(y) \Leftrightarrow$ long-run average costs are increasing in y

We further note that if the short-run fixed factors $\underline{x}_F$ are fixed at their long-run conditional factor demand for a given output, then

$$c^*(\underline{w}, y) = c_S^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y) \quad \forall \; y > 0 \Rightarrow$$

$$LMC(y^*) = SMC(y^*)$$

The argument is as follows: $c^*(\underline{w}, y) = c_S^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y) \quad \forall \; y > 0$

That means, we can take the total derivative on both sides. That is

$$LMC(y) = \frac{dc^*(\underline{w}, y)}{dy} = \frac{d}{dy} c_S^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)$$

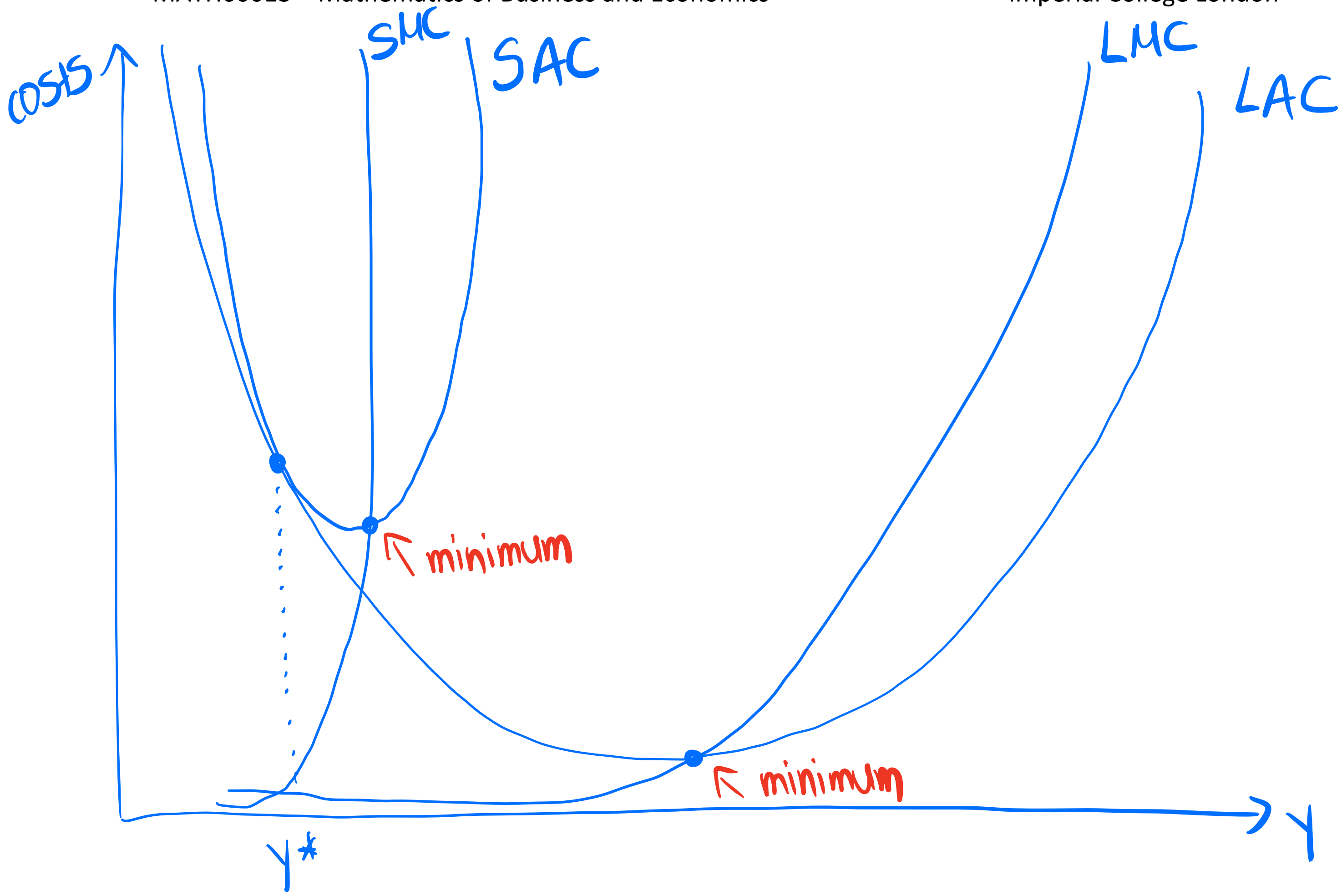
$$= \underbrace{\frac{\partial}{\partial \underline{x}_F} c_S^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)}_{0} \cdot \frac{\partial}{\partial y} \underline{x}_F^*(\underline{w}, y) + \underbrace{\frac{\partial}{\partial y} c_S^*(\underline{w}, \underline{x}_F^*(\underline{w}, y), y)}_{= SMC(y)}$$

The first summand is 0 since the function $\underline{x}_F \mapsto c_S^*(\underline{w}, \underline{x}_F, y)$ is minimised by $\underline{x}_F = \underline{x}_F^*(\underline{w}, y)$

So the first order condition implies that the partial derivative vanishes at this point.

With the second summand, one also needs to be cautious. This is only the SMC if $\underline{x}_F = \underline{x}_F^*(\underline{w}, y)$ which means that $y = y^* \Rightarrow LMC(y^*) = SMC(y^*)$.

The argument is very similar to the one in the Envelope Theorem.


costs
SMC
SAC
LMC
LAC
minimum
minimum
Y*
Y

Profit maximisation given minimised costs

We have now considered the choices that a firm must make to minimise its costs, given knowledge of its factor prices $\underline{w}$ and a given level of output y. As mentioned previously, we now consider how a firm should subsequently choose an optimal level of output y in order to maximise profits conditional on minimised costs.

To set up the profit-maximisation question in this conditional framework, we initially maintain the assumption of perfect competition; we also assume to begin with that we are operating in the short-run.

Recall that previously, profit maximisation was framed as a question of how much input to use, and that the output of the firm was specified by the production function f. Now, all of the firm's technical constraints are implicitly specified by the cost function.

We therefore reformulate our profit maximisation problem: we seek

$$\underset{y>0}{\operatorname{argmax}} \; py - c_S^*(\underline{w}, \underline{x}_F, y)$$

$$\max_{y>0} \; py - c_S^*(\underline{w}, \underline{x}_F, y)$$

First- and second-order conditions for the optimal level of output given minimised costs are given by:

$$\text{FOC: } \frac{\partial}{\partial y}\left(py - c_S^*(\underline{w}, \underline{x}_F, y)\right) = 0 \iff p = SMC(y)$$

$$\text{SOC: } \frac{\partial^2}{\partial y^2}\left(py - c_S^*(\underline{w}, \underline{x}_F, y)\right) \leq 0 \iff \frac{\partial SMC(y)}{\partial y} \geq 0$$

These conditions suggest that, in order to maximise profits, the output should be such that the corresponding short-run marginal cost is increasing and equal to the output price p.
(SOC) (FOC)

For a cost-minimising competitive firm, this specifies a relationship between the market-defined output price p and the quantity of output that the firm should provide – this is the short-run supply curve for an individual firm

Example 1:

Suppose that a firm's short-run cost function for a good is specified as

$$c_S^*(w_1, w_2, \underline{x}_F, y) = 2\sqrt{w_1 w_2}\, y^2 + FC(\underline{w}_F, \underline{x}_F)$$

If the market price for the good is £16 and each input costs the firm £4, how many units of the good should the firm produce in the short run, and what is their maximised profit if fixed costs are £12?

$p = £16$ $\quad$ $w_1 = w_2 = £4$

FOC: $SMC(y) = p \Rightarrow 4\sqrt{w_1 w_2}\, y = p \Rightarrow \hat{y} = \dfrac{p}{4\sqrt{w_1 w_2}} = \dfrac{\cancel{£}16}{\cancel{£}16} = 1$

SOC: $\dfrac{\partial}{\partial y} SMC(y) = 4\sqrt{w_1 w_2} \geq 0 \Rightarrow$ Maximum $\hat{y} = 1$

Maximum profit $= p\hat{y} - C_S^*(w_1, w_2, x_F, \hat{y}) = \cancel{£}16(1) - 2(\cancel{£}4)(1) - \cancel{£}12 = -£4$

In the short-run, i.e. when there are fixed costs, the most profitable position for a firm may be one that returns negative profit, as fixed costs will always require payment.

Example 2:

Consider the cost function

$$C_S^*(\underline{w}, x_F, y) = w_1 y^{1/2} + w_2 y^2 + FC(\underline{w}_F, x_F), \quad y > 0$$

What is the maximised profit here, when $w_1 = 2$, $w_2 = \frac{1}{2}$, and $p = 2$?

$w_1 = 2,\ w_2 = 1/2,\ p = 2$

FOC: $SMC(y) = \dfrac{w_1}{2} y^{-1/2} + 2w_2 y = p \Rightarrow$

$$y^{-1/2} + y = 2 \Rightarrow$$

$$\frac{1}{y^{1/2}} = 2 - y \Rightarrow$$

$$1 = 2y^{1/2} - y^{3/2} \Rightarrow$$

$$y^{3/2} - 2y^{1/2} + 1 = 0$$

set $y^{1/2} = x$: $x^3 - 2x + 1 = 0$

$\longrightarrow x_1 = 1$

$\longrightarrow x_2 = \dfrac{-1+\sqrt{5}}{2}$

$\longrightarrow x_3 = \dfrac{-1-\sqrt{5}}{2} < 0$, reject

$y_1 = x_1^2 = 1$

$y_2 = x_2^2 = \dfrac{3-\sqrt{5}}{2}$

SOC: $\dfrac{\partial^2 C_S^*(\underline{w}, x_F, y_1)}{\partial y^2} > 0$ and $\dfrac{\partial^2 C_S^*(\underline{w}, x_F, y_2)}{\partial y^2} < 0 \Rightarrow \boxed{\hat{y} = y_1 = 1}$

Maximum profit $= py_1 - w_1 y_1^{1/2} - w_2 y_1^2 - FC(\underline{w}_F, x_F) = \dots = -\dfrac{1}{2} - FC < 0$

So, as illustrated in Example 2, in some circumstances it may be preferable for a firm to go out of business rather than provide $y > 0$.

Indeed, we can generalise: it will be preferable to go out of business when

profit when $y=0$ ← $-\underline{w}_F \underline{x}_F^T > py - c_s^*(\underline{w}, \underline{x}_F, y) \iff$

$$0 > py - \underline{w}_V \underline{x}_V^*(\underline{w}, y)^T \iff$$

$$\underline{w}_V \underline{x}_V^*(\underline{w}, y)^T > py \overset{y>0}{\iff}$$

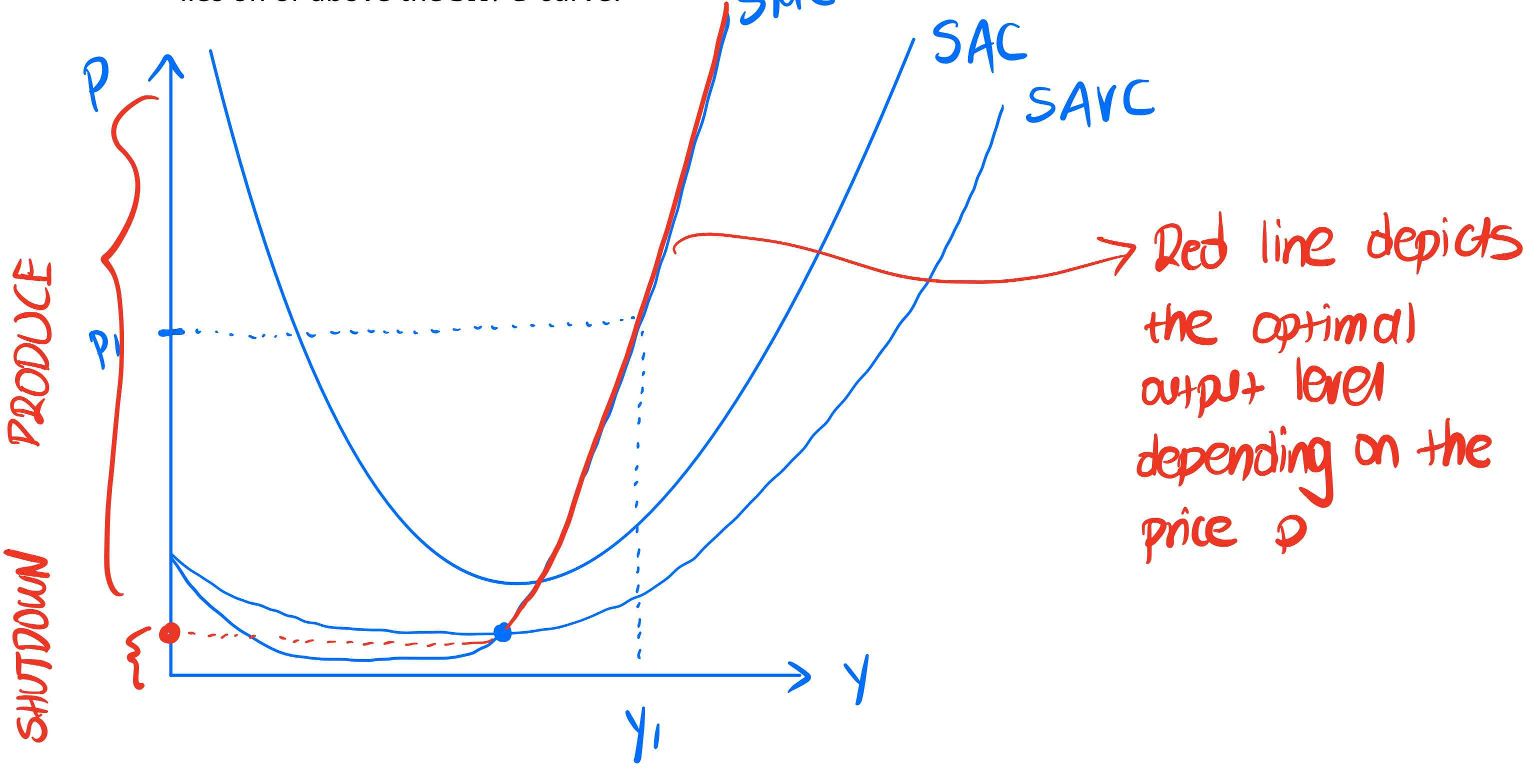
$$(*) \quad SAVC(y) = \frac{\underline{w}_V \underline{x}_V(\underline{w}, y)^T}{y} > p$$

This is known as the ***shutdown condition***; when satisfied, it is preferable for the firm to go out of business.
So we must refine our definition of the firm's chosen short-run supply. The competitive cost-minimising firm should choose a positive level of output y such that:

- $SMC(y) = p$;
- $SMC(y)$ is increasing in y;
- $SAVC(y) \leq p$ (converse of the shutdown condition

If no such $y > 0$ exists for the given p, then the firm should set $y = 0$.

These conditions are satisfied by the portion of the SMC curve that is increasing in y and that lies on or above the $SAVC$ curve:



In the long run, we have a very similar story. Neither the first- nor second-order conditions above explicitly require the costs to be dependent on fixed factors of production; these translate to the long-run scenario as would be expected. The long-run profit-maximising supply for a cost-minimising firm is given by y such that

- $LMC(y) = p$ (FOC)
- $LMC(y)$ increasing in y (SOC)
- $LAC(y) \leq p$ (converse of the shutdown condition)

Once more, if no such $y > 0$ exists for the given p, then the firm should choose to go out of business.

Profit maximisation for a noncompetitive firm

To contrast, we consider the profit maximisation problem for a cost-minimising monopolist. Whilst monopolists have more control over output prices than in a competitive market, they cannot choose price and output independently of one another; they must respect the market demand for their product. We therefore assume that the monopolist chooses the amount of output to provide, y, and the output price is determined according to the market demand for this output, i.e. as a function of y, $p(y)$.

The function $p(y)$ is the inverse of the market's demand function and is referred to as the inverse demand function "facing the firm"; we note that it may be dependent on other determinants, but assume these to be held constant in our analysis.

To maximise profits, we therefore seek

$$\underset{y\geq 0}{\operatorname{argmax}}\{p(y)y - c_S^*(\underline{w},y)\}$$

First- and second-order conditions for finding a profit-maximising position for a monopolist facing an inverse demand function are therefore given by

$$\frac{\partial}{\partial y}\left(p(y)y - c_S^*(\underline{w},y)\right) = 0 \quad \Rightarrow \quad \frac{\partial p(y)}{\partial y}y + p(y) = SMC(y) \qquad (FOC)$$

$$\frac{\partial^2}{\partial y^2}\left(p(y)y - c_S^*(\underline{w},y)\right) \leq 0 \quad \Rightarrow \quad \frac{\partial^2 c_S^*(\underline{w},\underline{x}_F,y)}{(\partial y)^2} \geq \frac{\partial^2 p(y)}{(\partial y)^2}y + 2\frac{\partial p(y)}{\partial y} \qquad (SOC)$$

We can rearrange the FOC:

$$p(y)\Bigg[\underbrace{1 + \frac{\partial p(y)}{\partial y}\cdot\frac{y}{p(y)}}_{1+\frac{1}{\varepsilon_D(y)}}\Bigg] = SMC(y)$$

where $\varepsilon_D(y) := \varepsilon_D(p(y))$ is the price elasticity of demand facing the monopolist.

Recall that $p(y) = D^{-1}(y)$ s.t. $y = D(p(y))$. Taking the derivative wrt y:

$$p'(y) = \frac{1}{D'(p(y))}$$

Since $\varepsilon_D(y) < 0$ and $SMC \geq 0$, we must have $|\varepsilon_D(y)| \geq 1$

$\rightarrow$ In order to have a profit-maximising position, we need to have elastic demand.

Example:

Consider the monopolist faced with a linear inverse demand

$$p(y) = a_1 - a_2 y \qquad a_1, a_2 > 0$$

and Cobb-Douglas variable costs in the short term

$$c_S^*(\underline{w}, \underline{x_F}, y) = 2\sqrt{w_1 w_2}\, y^2 + FC(\underline{w_F}, \underline{x_F}).$$

What is the maximum profit that this monopolist can achieve?

$$\frac{1}{\varepsilon_D(y)} = \frac{\partial p(y)}{\partial y} \cdot \frac{y}{p(y)} = -a_2 \frac{y}{a_1 - a_2 y} = -\frac{a_2 y}{a_1 - a_2 y}.$$

$$|\varepsilon_D(y)| \geq 1 \Rightarrow \frac{a_1}{a_2 y} - 1 \geq 1 \Rightarrow y \leq \frac{a_1}{2a_2}$$

so any profit-maximising level of output must be below or equal to $\frac{a_1}{2a_2}$.

FOC: $$p(\hat{y})\left[1 + \frac{1}{\varepsilon_D(\hat{y})}\right] = SMC(\hat{y})$$

$$(a_1 - a_2\hat{y})\left[1 + \left(-\frac{a_2\hat{y}}{a_1 - a_2\hat{y}}\right)\right] = 4\sqrt{w_1 w_2}\,\hat{y} \Rightarrow \dots$$

$$\hat{y} = \frac{a_1}{2a_2 + 4\sqrt{w_1 w_2}} \leq \frac{a_1}{2a_2} \text{ as required.}$$

Evaluating the SOC verifies that $\hat{y}$ is a maximum.

$$\text{Maximum profit} = p(\hat{y})\cdot\hat{y} - c_S^*(\underline{w}, \underline{x_F}, y) = \dots$$

$$= \frac{a_1^2}{4a_2 + 8\sqrt{w_1 w_2}} - FC(\underline{w_F}, \underline{x_F})$$

We can see from this example that it is also possible for profit-maximising monopolists to experience losses in the short-run; this is not a phenomenon unique to competitive markets.

The above optimisation assumes that $y > 0$. Just as for competitive firms, however, we note that the profit-maximising (loss-minimising) position for a monopolist may be to go out of business, i.e. to set $y = 0$. This happens when the losses incurred by setting output according to the above first- and second-order conditions are greater than the fixed costs, i.e. when

$$SAVC(y) > p(y)$$

We also note that, as for competitive firms, the extension to the long-run is trivial. For a cost-minimising monopolist, the long-run profit-maximising output y will satisfy the following conditions:

- $p(y)\left[1 + \frac{1}{\varepsilon_D(y)}\right] = LMC(y)$ (FOC)
- $\frac{\partial^2 c^*(\underline{w}, y)}{\partial y^2} > \frac{\partial^2 p(y)}{\partial y^2} \cdot y + 2\frac{\partial p(y)}{\partial y}$ (SOC)
- $AC \leq p(y)$ (converse of the shutdown condition)

Theory of the Consumer

We now focus on the theory of the consumer, where we will formalise the notion of consumer preferences and show how optimal behaviour of the consumer with respect to their preferences will lead to a specification of the demand function.
In the course of our analysis, we will see a lot of similarities and analogies to the Theory of the Firm.

Preferences & Utility

We start by considering the goods consumed by a consumer.

Define the ***consumption bundle*** for a particular consumer to be the quantities of a collection of goods that the consumer is willing to consume:

$$\underline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_{\geq 0}.$$

The set of possible consumption bundles is referred to as the ***consumption set***; this is usually taken to be some *closed and convex* set

$$X \subseteq \mathbb{R}^n_{\geq 0}.$$

Consumers are assumed to have preferences between bundles $\underline{x}, \underline{x}' \in X$:

- $\underline{x} \preccurlyeq \underline{x}'$ means that the consumer has a preference for bundle $\underline{x}'$ over bundle $\underline{x}$.
-
- $\underline{x} \prec \underline{x}'$ means that the consumer has a *strict* preference for $\underline{x}'$ over $\underline{x}$.

  $\underline{x} \prec \underline{x}' \iff \underline{x} \preccurlyeq \underline{x}'$ but not $\underline{x}' \preccurlyeq \underline{x}$
- $\underline{x} \sim \underline{x}'$ denotes indifference between $\underline{x}$ and $\underline{x}'$.

  $\underline{x} \sim \underline{x}' \iff \underline{x} \preccurlyeq \underline{x}'$ and $\underline{x}' \preccurlyeq \underline{x}$

[Mathematically speaking, the preference relation "$\preccurlyeq$" is a subset of the cartesian product $X \times X$.]

We are working under the condition that the preference relation satisfies the three axioms of a **complete weak order** on X. That is

- Completeness: $\forall\, \underline{x}, \underline{x}' \in X$: $\underline{x} \preccurlyeq \underline{x}'$ or $\underline{x}' \preccurlyeq \underline{x}$
- Reflexivity: $\forall\, \underline{x} \in X$: $\underline{x} \preccurlyeq \underline{x}$
- Transitivity: $\forall\, \underline{x}, \underline{x}', \underline{x}'' \in X$: if $\underline{x} \preccurlyeq \underline{x}'$ and $\underline{x}' \preccurlyeq \underline{x}''$ then $\underline{x} \preccurlyeq \underline{x}''$

Beware that reflexivity actually follows from completeness.

In addition, the following assumptions are *useful* but not necessary:

Continuity

$\forall\ \underline{x} \in X$ the sets $\{\underline{x}' \in X : \underline{x} \preccurlyeq \underline{x}'\}$ and $\{\underline{x}' \in X : \underline{x}' \preccurlyeq \underline{x}\}$ are closed sets.

Weak / Strong Monotonicity

$$\underline{x} \leq \underline{x}' \quad \Longrightarrow \quad \underline{x} \preccurlyeq \underline{x}' \quad \text{(weak)}$$

$$\underline{x} \leq \underline{x}' \ \ and \ \ \underline{x} \neq \underline{x}' \quad \Longrightarrow \quad \underline{x} \prec \underline{x}' \quad \text{(strong)}$$

Local nonsatiation

$\forall\ \underline{x} \in X\ \forall\ \varepsilon > 0:$

$\exists\ \underline{x}' \in X$ with $\|\underline{x} - \underline{x}'\| < \varepsilon$ and $\underline{x} \prec \underline{x}'$

(Strict) Convexity

$\forall\ \underline{x}, \underline{x}', \underline{x}'' \in X$ with $\underline{x} \preccurlyeq \underline{x}'$ and $\underline{x} \preccurlyeq \underline{x}''$, then

$$\underline{x} \preccurlyeq t\underline{x}' + (1-t)\underline{x}'' \quad \forall\ t \in [0,1]$$

Strict convexity:

$\forall\ \underline{x}, \underline{x}', \underline{x}'' \in X$ with $\underline{x} \preccurlyeq \underline{x}'$ and $\underline{x} \preccurlyeq \underline{x}''$,

$\underline{x}' \neq \underline{x}''$ $\quad \underline{x} \prec t\underline{x}' + (1-t)\underline{x}'' \quad \forall\ t \in (0,1)$

Note – we have not yet used the symbols $\succcurlyeq$ or $\succ$; we can use this as would be expected, i.e.

$$\underline{x} \preccurlyeq \underline{x}' \iff \underline{x}' \succcurlyeq \underline{x}$$

but it is no more than a notational convenience.

How does a consumer decide between bundles in some subset of X? How do we judge the suitability, or usefulness, of a consumption bundle $\underline{x}$? More to the point, how can we, as economists, model the unobserved preference allocation of consumers?

It is useful to model consumer preferences by a ***utility function***, which we define to be a real mapping $u: X \to \mathbb{R}$.
We say that u ***represents the preference relation*** $\preccurlyeq$ if

$$\forall\ \underline{x}', \underline{x} \in X : \quad u(\underline{x}') \leq u(\underline{x}) \iff \underline{x}' \preccurlyeq \underline{x}$$

We will only consider ordinal utilities!

- If only the ordering imposed by a utility function is relevant, one speaks of an **ordinal utility**. If u is an ordinal utility, any strictly increasing transformation of u represents the same preferences., e.g. $u_1(x_1, x_2) = x_1 x_2$ $\quad u_2(x_1, x_2) = \log u_1 = \log(x_1 x_2)$
- If one wants to compare different utility differences, say

  $$|u(\underline{x}') - u(\underline{x})| \quad \text{and} \quad |u(\underline{x}'') - u(\underline{x})|$$

  one speaks of a **cardinal utility.** Cardinal utilities are in general only preserved by affine and increasing transformations.

Existence of an (ordinal) utility function:

Suppose a consumption set X is imbued with a preference relation that is complete, transitive, continuous and strongly monotonic. Then there exists a continuous utility function $u : X \to \mathbb{R}$ that represents this preference relation.

Note – the assumption of strong monotonicity can be dropped, though the proof is more complex. – Debreu's Theorem (1954)

Proof:

Outline:

- We will consider bundles of goods that contain the same amount of each good, i.e. 'homogeneous' bundles;
- We will show that if, for every $\underline{x} \in X$, there exists a homogeneous bundle to which the consumer is indifferent, then the level of the homogeneous bundle can be taken as an appropriate utility function, i.e. one that preserves the ordering of $\succcurlyeq$;
- We will then show that such a homogeneous bundle exists and is unique.

Let $\underline{e} = \{1, \dots, 1\}$ be a vector of n ones.

Suppose that for any consumption bundle $\underline{x} \in X$ there exists $u(\underline{x}) \in \mathbb{R}$ s.t. $u(\underline{x}) \cdot \underline{e} \sim \underline{x}$.

We show that such a number represents the preference relation.

Indeed, for any $\underline{x}, \underline{x}' \in X$:

$$\begin{aligned} u(\underline{x}) > u(\underline{x}') &\Rightarrow u(\underline{x}) \cdot \underline{e} > u(\underline{x}') \cdot \underline{e} \\ &\Rightarrow u(\underline{x}) \cdot \underline{e} \succ u(\underline{x}') \cdot \underline{e} \text{ (monotonicity)} \\ &\Rightarrow \underline{x} \succ \underline{x}' \text{ (transitivity)} \end{aligned}$$

Similarly, we can show that $u(\underline{x}) \leq u(\underline{x}') \Rightarrow \underline{x} \preceq \underline{x}'$.

So $u(\underline{x})$ maintains the preference relation.

<u>Existence of $u(\underline{x})$:</u> Let $\underline{x} \in X \subseteq \mathbb{R}^n_{\geq 0}$.

- Define $B = \{t \in \mathbb{R} \mid t\underline{e} \succsim \underline{x}\}$ and

  $W = \{t \in \mathbb{R} \mid t\underline{e} \preceq \underline{x}\}$

- $(\max_i x_i)\, \underline{e} \geq \underline{x} \Rightarrow (\max_i x_i)\, \underline{e} \succsim \underline{x} \Rightarrow (\max_i x_i) \in B$
- $0 \cdot \underline{e} \preceq \underline{x} \Rightarrow 0 \in W$

$\Big\}$ B and W are non-empty sets

- By continuity of $\preceq$, the sets B and W are closed. That means they have an upper and lower bound, which are contained in these sets respectively.

- Set $t^* := \inf B \in B$. Let $t_n = t^* - \frac{1}{n}$

  • $t_n < t^* \Rightarrow t_n \notin B$

  $\Rightarrow t_n \cdot \underline{e} \prec \underline{x}$

  $\Rightarrow t_n \in W$

  • $t_n \to t^*$. Moreover, W is a closed set $\Rightarrow t^* \in W$.

- Since $t^* \in B \cap W$, $t^* \underline{e} \sim \underline{x}$.

- We set $u(\underline{x}) := t^*$, and therefore exists.

<u>Uniqueness</u>

• Suppose that for some $\underline{x} \in X$ there are $u_1(\underline{x})$ and $u_2(\underline{x})$ s.t. $u_1(\underline{x}) \cdot \underline{e} \sim \underline{x}$ and $u_2(\underline{x}) \cdot \underline{e} \sim \underline{x}$.

- By transitivity: $u_1(\underline{x}) \underline{e} \succsim \underline{x} \succsim u_2(\underline{x}) \cdot \underline{e}$

  $\Rightarrow u_1(\underline{x}) \underline{e} \succsim u_2(\underline{x}) \cdot \underline{e}$

  $\Rightarrow u_1(\underline{x}) \geq u_2(x)$ (using monotonicity)

- Similarly, $u_1(x) \leq u_2(x) \Rightarrow u_1(\underline{x}) = u_2(x)$.

Properties of a utility function

If the underlying preferences are complete, transitive, continuous and (strictly) monotone, the corresponding utility function will be continuous and (strictly) monotone.

If the preferences are (strictly) convex, the utility function is (strictly) quasi-concave.

Note that a function $f: X \to \mathbb{R}$ is strictly quasi-concave if for all $\underline{x}, \underline{y} \in X$, $x \neq y$ and for all $t \in (0,1)$:

$$f\left((1-t)\underline{x} + t\underline{y}\right) > \min\{f(\underline{x}), f(\underline{y})\}$$

Substitution in demand

Suppose the availability of good i drops, such that x_i must decrease. In order to preserve the same level of utility in their overall consumption bundle, consumers will want to compensate by replacing with a separate good. By how much should the consumer alter x_j such that the utility remains constant?

[This is analogous problem to that of technical substitution.]

Indeed, we define the ***marginal rate of substitution*** (MRS) to be the rate of change of good j with respect to the change in good i:

$$MRS_{i,j}(\underline{x}) = -\frac{\frac{\partial u(\underline{x})}{\partial x_i}}{\frac{\partial u(\underline{x})}{\partial x_j}}$$

where we also define

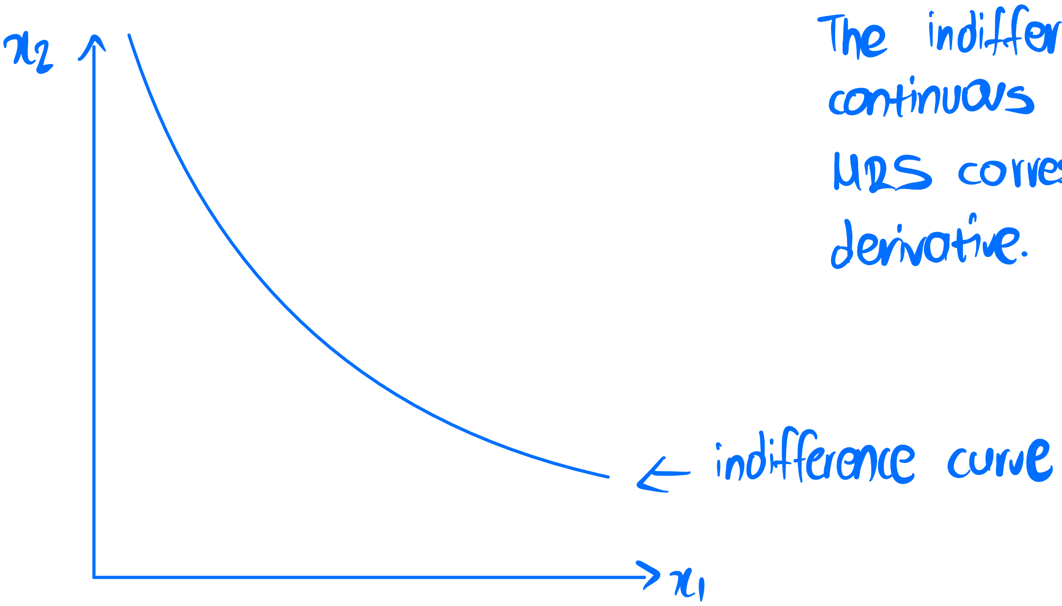
$$MU_i(\underline{x}) = \frac{\partial u(\underline{x})}{\partial x_i}$$

(!) $x_i \to 0$? $x_i \to \infty$?

to be the ***marginal utility*** with respect to good i.

The MRS is, of course, the consumer-side analogue to the MRTS (marginal rate of technical substitution. One can check that the $MRS_{i,j}$ is indeed invariant under a strictly monotonic transformation of the utilities.

Just as it is often useful to consider a graphical representation of a firm's economic and technological capabilities, it can be useful to graphically represent consumer preferences. As a demand-side analogue to the isoquant, we define the ***indifference curve*** to be a level set of the utility function:



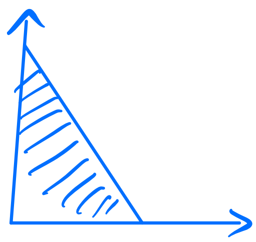
The indifference curve is continuous and convex and MRS corresponds to its derivative.

Budget Restraints, Utility Maximisation and Demand

In practice, consumers can't simply pick their most preferred bundle – ∃ budget restraints. A fundamental assumption underlying consumer-side economic analysis is that the consumer will choose to purchase the most preferred consumption bundle from the set of all *affordable* bundles.

Represent the set of all affordable bundles by the **budget set**:

$$B = B_{\underline{p},m} = \{\underline{x} \in X : \underline{p}\,\underline{x}^T \leq m\} \subseteq X$$



At the heart of consumer choice, then, is the problem of finding the most preferred bundle $\underline{x} \in B$.

This is the problem of finding $\underset{\underline{x} \in B_{\underline{p},m}}{\operatorname{argmax}}\, u(\underline{x})$

A solution to this problem will exist so long as u is continuous and B is closed and bounded…

– this is guaranteed if $\underline{p} > 0$

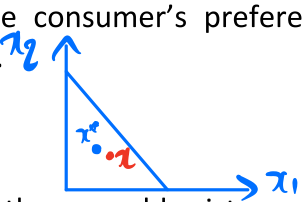
Denote the constrained utility-maximising bundle $\underline{x}^* \in B$:

- $\underline{x}^*$ will be independent under a strictly increasing transformation of utility function.
- $\underline{x}^*$ will, in general, be dependant both on prices $\underline{p}$ and on the budget m.
- $\underline{x}^*$ is homogeneous of degree zero *jointly* in prices and budget.

How can we find $\underline{x}^*$? → Langrangian (constraint given in terms of equalities)

↓ Kuhn-Tucker (constraint given in terms of inequalities)

If we make some reasonable regularity assumptions about the consumer's preference ordering $\preccurlyeq$, we can simplify our constrained optimisation problem.



Assume *local nonsatiation* and suppose $\underline{x}^* = \operatorname*{argmax}_{\underline{x}\in B} u(\underline{x})$:

- If $\underline{p}\,\underline{x}^{*\top} < m$, that means $\underline{x}^*$ is in the interior of B, then there would exist some $\underline{x}$, close enough to $\underline{x}^*$, such that both $\underline{p}\,\underline{x}^\top < m$ and (by nonsatiation) $\underline{x} \succ \underline{x}^*$.
- This would imply that $\underline{x}^*$did not maximize $u(x)$, and so we have a contradiction.

$$\Longrightarrow \underline{p}\,\underline{x}^{*\top} \not< m$$
$$\Longrightarrow \underline{p}\,\underline{x}^{*\top} = m.$$

Thus, since we assume local nonsatiation, we need only seek $\operatorname*{argmax}_{\underline{x}\in\partial B} u(\underline{x})$. We can address this using the Lagrangian!

Some economists call the fact that utilities are maximised only if people spend all their money **Walras' Law.**

Example: Consider the consumer with utility function $u(x_1,x_2) = x_1^{\alpha} x_2^{1-\alpha}$, $\alpha \in (0,1)$.

We seek $\operatorname*{argmax}_{\underline{x}\in X} u(\underline{x})$ s.t. $\underline{p}\,\underline{x}^\top = m$

$$\mathcal{L}(x_1,x_2,\lambda) = x_1^{\alpha} x_2^{1-\alpha} - \lambda(p_1x_1 + p_2x_2 - m)$$

FOC:
$$\left.\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \longrightarrow \text{constraint} \\[2ex] \dfrac{\partial \mathcal{L}}{\partial x_i} = 0, \quad i = 1,2 \end{array}\right\} \Rightarrow$$

$$p_1x_1 + p_2x_2 = m \quad (1)$$
$$\alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \quad (2)$$
$$(1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0 \quad (3)$$

$$(2) \div (3) \overset{\lambda \neq 0}{\Longrightarrow} \frac{\alpha}{1-\alpha}\left(\frac{x_2}{x_1}\right) = \frac{p_1}{p_2}$$

Express x_1 in terms of x_2 and plug it into the constraint (1)...

$$x_2 = \frac{1-\alpha}{\alpha}\left(\frac{p_1}{p_2}\right)\left(\frac{m - p_2 x_2}{p_1}\right) \Rightarrow x_2^*(p_1, p_2, m) = \frac{m(1-\alpha)}{p_2}$$

By symmetry: $x_1^*(p_1, p_2, m) = \dfrac{m\alpha}{p_1}$

Verify properties $\rightarrow$ homogeneity (hom. of degree 0 in m and p)

$\uparrow$ with m

$\downarrow$ with its own price

* Cobb-Douglas u.f. (only), x_1 is independent from p_2

x_2 — " — p_1

working with $u_1(\underline{x}) = \log u(\underline{x}) \rightarrow x^*$

A note on second-order conditions for the Lagrangian:

In order for the Lagrangian $\mathcal{L}$ to be maximised, we require negative semidefiniteness of the matrix of second derivatives of $\mathcal{L}$ with respect to each of the variables (necessary condition)

This matrix is known as the '**bordered Hessian**'; in the current context, the Hessian refers to the matrix of second derivatives of the utility function. Requiring the bordered Hessian to be negative semidefinite is the same as requiring the Hessian to be negative semidefinite, subject to a linear constraint:

$$\underline{h}\, \nabla^2 u(\underline{x}^*)\, \underline{h}^T \leq 0 \quad \forall \underline{h} \in \mathbb{R}^n \text{ s.t. } \nabla u(\underline{x}^*)\, \underline{h}^T = 0$$

$$\text{where } \nabla^2 u(\underline{x}^*) = \left(\frac{\partial^2 u(\underline{x}^*)}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,n}$$

This is necessary for the fact that the utility function u is *locally* quasi-concave.

That means that there is a neighbourhood U containing $\underline{x}^*$ s.t. u is quasi-concave on U.

A *sufficient* second order condition for local quasi-concavity is that the utility function is *strictly locally quasi-concave.* A sufficient condition for that is

$$\underline{h}\, \nabla^2 u(\underline{x}^*)\, \underline{h}^T < 0 \quad \forall \underline{h} \in \mathbb{R}^n,\ \underline{h} \neq 0, \text{ s.t. } \nabla u(\underline{x}^*)\, \underline{h}^T = 0$$

The choice of the consumption bundle that maximises the consumer's constrained utility function will be exactly the bundle that the consumer demands; this is unsurprisingly referred to as the ***demanded bundle*** or ***demand function***,

$$\underline{x}^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \qquad \underline{x}^*\left(\underline{p}, m\right) = \operatorname*{argmax}_{\underline{x} \in \partial B_{\underline{p},m}} u\left(\underline{x}\right).$$

We call this the Marshallian demand function (or uncompensated demand function).

Note that we have discussed the existence of the argmax. However, it is *per se* not clear whether the argmax is unique, that means, whether the maximum is attained at a single point over ∂B. This can be guaranteed if the underlying preferences are strictly convex (and prices are strictly positive).

Moreover, the function $\underline{x}^*(\underline{p}, m)$ is homogeneous of degree 0 in $(\underline{p}, m)$.

We also note that, faced with a set of goods with prices $\underline{p}$, the maximum utility achievable with a given budget m is known as the ***indirect utility function***:

$$v : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad v\left(\underline{p}, m\right) = u\left(\underline{x}^*\left(\underline{p}, m\right)\right) = \max_{\underline{x} \in B_{\underline{p},m}} u(\underline{x})$$

This indirect utility function is itself a quantity of interest, and we note some of its key properties here:

- Nonincreasing in $\underline{p}$:

$$\underline{p}' \geq \underline{p} \Rightarrow v\left(\underline{p}', m\right) \leq v\left(\underline{p}, m\right)$$

...and nondecreasing in m:

Note: If local nonsatiation, then v is strictly increasing in m.

$$m' \geq m \Rightarrow v\left(\underline{p}, m'\right) \geq v\left(\underline{p}, m\right)$$

- Homogeneous of degree 0 in $\left(\underline{p}, m\right)$:

$$v\left(t\underline{p}, tm\right) = v\left(\underline{p}, m\right) \quad \forall t > 0$$

- Quasi-convex in $\underline{p}$:

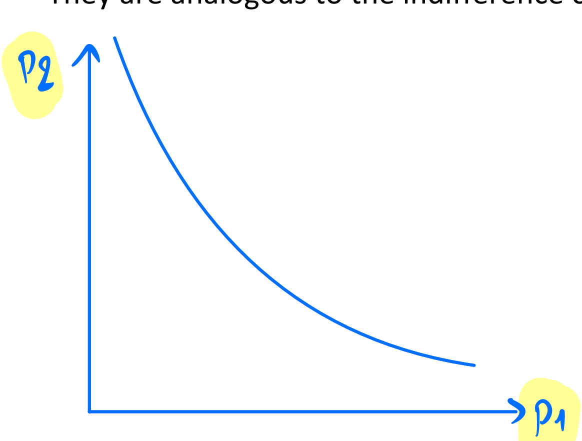
$\left\{\underline{p} \in \mathbb{R}^n_{\geq 0} : v\left(\underline{p}, m\right) \leq k\right\}$ is a convex set for all $k \in \mathbb{R}$., $m \geq 0$

- Continuous at all $\underline{p} \gg 0, m > 0$.

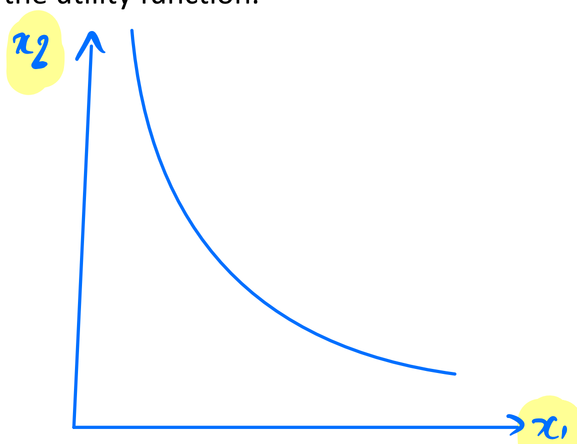
The indirect utility function v is often illustrated using so-called price indifference curves. These are the level sets of the indirect utility function with a *fixed budget* m. That is

$$\left\{ \underline{p} \in \mathbb{R}^{n}_{\geq 0} : v(\underline{p}, m) = k \right\}, \ k \in \mathbb{R}, \ m \geq 0$$

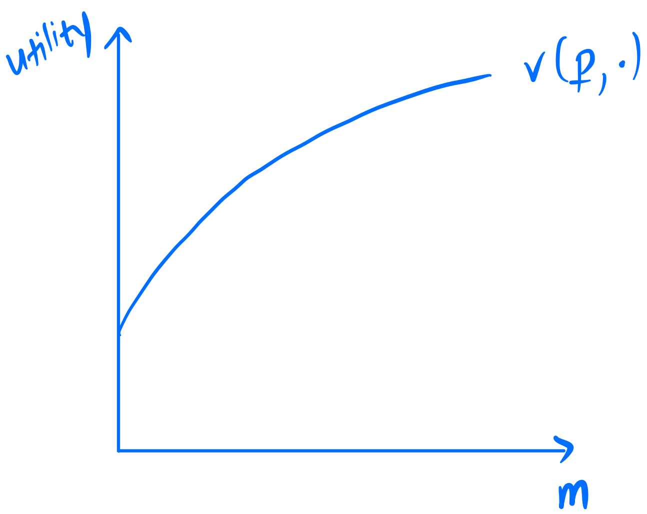
They are analogous to the indifference curves of the utility function.



price indifference curve
$\{(p_1, p_2) : v(p_1, p_2, m) = k\}$



indifference curve
$\{(x_1, x_2) : u(x_1, x_2) = k\}$



A direct consequence of the local nonsatiation assumption of the underlying preferences is that for fixed $\underline{p}$, the indirect utility function $v\left(\underline{p},\ \cdot\right)$ is strictly increasing in m.

Therefore, $v\left(\underline{p},\ \cdot\right)$ is injective and can be inverted on its image. Denote this image with

$$U_p = \{ v\left(\underline{p}, m\right) : m \geq 0\}.$$

Then we define the ***expenditure function***

$$e(\underline{p},\cdot): U_p \to [0,\infty) \quad u \longmapsto e(\underline{p},u) \text{ s.t. } u = v(\underline{p}, e(\underline{p},u))$$

* Notation: $u(\underline{x})$ utility function vs u level of utility

The expenditure function provides the minimum level of income required to obtain a given level of utility at prices $\underline{p}$. Note that $e\left(\underline{p}, u\right)$ can also be obtained as the solution to the optimisation problem

Find $\min_{\underline{x}} \underline{p}\underline{x}^T$ s.t. $u(\underline{x}) \geq u$.

The consumer's expenditure function is simply the demand-side analogue of the firm's cost function.

They share the same properties:

- nondecreasing in $\underline{p}$
- homogeneous of degree 1 in $\underline{p}$
- concave in $\underline{p}$
- continuous in $\underline{p}$ for all $\underline{p} > 0$

The dual quantity to the expenditure function is the ***Hicksian demand*** (sometimes referred to as the *compensated demand*)

$$\underline{x}_H^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R} \to \mathbb{R}^n_{\geq 0}, \qquad \left(\underline{p}, u\right) \mapsto \underline{x}_H^*\left(\underline{p}, u\right) = \underset{\underline{x} \in u^{-1}([u,\infty))}{\operatorname{argmin}} \underline{p}\underline{x}^T$$

Recall that, on the firm side, for a specified level of output, the cost-minimising combination of production inputs can be found via Shephard's Lemma. We can also apply this result in the current scenario, yielding an expression for the expenditure-minimising consumption bundle in terms of prices and desired utility level:

$$x_{H,i}^*\left(\underline{p}, u\right) = \frac{\partial e\left(\underline{p}, u\right)}{\partial p_i}.$$

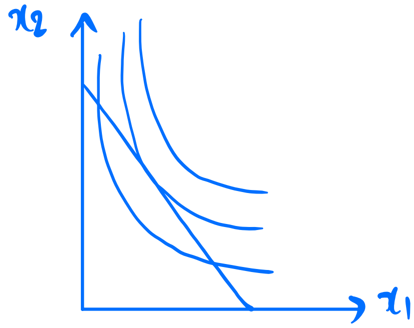
The ***Hicksian demand function*** is formally the same as the conditional factor demand function on the supply side.

Note that, when we refer to the demand function without qualification, it is assumed to be the Marshallian demand.

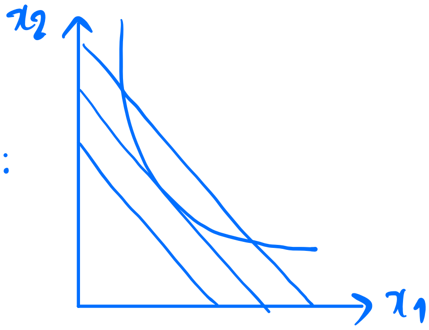
Unlike the Marshallian demand, the Hicksian demand function is not observable; indeed, it depends on utility, which is itself unobservable. Nonetheless, under some of the **usual regularity assumptions,** the Hicksian and Marshallian demands satisfy the following identities: For all $\underline{p} > 0$, $m \geq 0$:

- $e\left(\underline{p}, \overbrace{v\left(\underline{p}, m\right)}^{u}\right) = m$
- $v\left(\underline{p}, \overbrace{e\left(\underline{p}, u\right)}^{m}\right) \equiv u$
- $x^*_{H,i}\left(\underline{p}, \overbrace{v\left(\underline{p}, m\right)}^{u}\right) = x_i^*(\underline{p}, m)$
- $x^*_i\left(\underline{p}, \overbrace{e\left(\underline{p}, u\right)}^{m}\right) = x^*_{H,i}(\underline{p}, u)$

M:



H:



The Slutsky equation

For economists, it is important to understand how consumers react to changes in the economic environment. For instance, we can consider how the optimal choice of consumption bundle $\underline{x}^*\left(\underline{p}, m\right)$ will change with respect to the price vector $\underline{p}$. *The **Slutsky equation*** states that the total effect of a change in demand of a good when the price of a good is changed can be decomposed into a ***substitution effect*** and an ***income effect.***

- the substitution effect is the change in the demanded bundle resulting from a change in the optimal balance of goods whilst keeping the level of utility fixed.
- the income effect is the change in the magnitude of the optimally balanced bundle, due to the increase/ decrease in purchasing power.

Theorem: Under the usual regularity conditions, we have that

$$\frac{\partial x_j^*(\underline{p},m)}{\partial p_i} = \frac{\partial x_{H,j}(\underline{p},v(\underline{p},m))}{\partial p_i} - \frac{\partial x_j^*(\underline{p},m)}{\partial m} \cdot x_i^*(\underline{p},m)$$

Total effect = Substitution effect + Income effect

for all $\underline{p} \gg 0, m > 0$ and for all $i, j \in \{1, \dots, n\}$.

Remark: Sometimes, notation can be quite misleading. Especially when considering the Slutsky equation it is crucial not to mix partial derivatives and total derivatives. A partial derivative is an operator mapping a (differentiable) function to its derivative which is again a function. In contrast, a total derivative needs a free variable in an equation and takes the derivative with respect to this equation.

Example $f: \mathbb{R}^2 \to \mathbb{R} \quad (x_1, x_2) \mapsto f(x_1,x_2) = x_1^3 + x_2^5$

$\frac{\partial}{\partial x_2} f$ is again a function $\mathbb{R}^2 \to \mathbb{R}$.

Actually, it is defined as $(x_1,x_2) \mapsto 5x_2^4$.

We can evaluate this function for arbitrary pairs $(a,b) \in \mathbb{R}^2$.

$$\left(\frac{\partial}{\partial x_2} f\right)(a,b) = 5b^4 \qquad \frac{\partial}{\partial x_2} f(2,1) = 5$$

On the other hand,

$$\frac{d}{dx_2} f(a,b) = 0 \qquad \frac{d}{dx_2} f(x_1,x_2) = 5x_2^4$$

Indeed, $f(a,b)$ is constant in x_2. We see that actually, the operator $\partial/\partial x_2$ indicates that we have to take the partial derivative wrt the second argument where usually x_2 stands. But the denomination can be quite misleading. It's better to indicate the argument, that is ∂_2, instead of $\partial/\partial x_2$.

In the light of this discussion, the Slutsky equation takes the form:

n goods (n prices) budget

$$\partial_i x_j^*(p,m) = \partial_i x_{H,j}^*(p, v(p,m)) - \partial_{n+1} x_j^*(p,m)\, x_i^*(p,m)$$

Proof of Slutsky's Equation:

Let $p > 0$, $m > 0$. For any $u \in U_p$ we have the identity:

$$x_{H,j}^*(p,u) = x_j^*(p, e(p,u)).$$

$$\partial_i x_{H,j}^*(p,u) = \frac{d}{dp_i} x_{H,j}^*(p,u) = \frac{d}{dp_i} x_j^*(p, e(p,u)).$$

$$= \partial_i x_j^*(p, e(p,u)) + \partial_{n+1} x_j^*(p, e(p,u)) \cdot \underbrace{\partial_i e(p,u)}_{x_{H,i}^*(p,u)}$$

We set $u = v(p,m)$. Then:

$$\partial_i x_{H,j}^*(p, v(p,m)) =$$

$$\partial_i x_j^*(p, \underbrace{e(p, v(p,m))}_{m}) + \partial_{n+1} x_j^*(p, \underbrace{e(p, v(p,m))}_{m}) \cdot \underbrace{x_{H,i}^*(p, v(p,m))}_{x_i^*(p,m)}$$

Rearranging yields the assertion. $\square$

It is clear from the Slutsky equation that the income effect plays a major part in determining how the demand for a set of goods will react to changes in their prices. For firms, consumers and economists alike, then, it is important to ascertain how the demand of certain goods will react to changes in consumer budget.

Indeed, economists class goods according to the manner in which they react to changes in consumer income:

- For ***normal goods***, an increase in income will result in an increase in demand;

$$\partial_{n+1} x_j^*(\underline{p},m) = \frac{\partial}{\partial m} x_j^*(\underline{p},m) \geq 0$$

- For ***inferior goods***, an increase in income will result in a decrease in demand.

$$\frac{\partial}{\partial m} x_j^*(\underline{p},m) < 0$$

It is also worth noting the different subclasses of normal goods: suppose we have an increase in consumer income m...

- ...for ***luxury goods***, demand will increase more than proportionally to income; (IED > 1)
- ...for ***necessary goods***, demand will increase less than proportionally (IED ∈ (0,1))
- ...and if demand increases proportionally to income, the consumer is said to have ***homothetic preferences*** for the set of goods under consideration. (IED = 1)

Finally, we note that goods can also be classified according to how changes in price impact their consumer demand:

- For ***ordinary goods***, a decrease in price will lead to an increase in their demand;

$$\frac{\partial}{\partial p_j} x_j^*(\underline{p}, m) \leq 0$$

- For ***Giffen goods***, a decrease in price will lead to a decrease in demand

$$\frac{\partial}{\partial p_j} x_j^*(\underline{p},m) > 0$$

e.g. staple food

!

That means our previously stated law of demand only holds for ordinary goods, but not for Giffen goods.

What is an example for a Giffen good? Some theoretical considerations can help us finding necessary conditions for Giffen goods. In particular, the Slutsky equation helps to establish a relation between ordinary and normal goods on the one hand side, as well as Giffen and inferior goods on the other side.

C-D utility function (pp. 48-49): $x_1^* = \frac{m(1-\alpha)}{p_2}$, $x_2^* = \frac{m\alpha}{p_1}$

$\frac{\partial x_1}{\partial m} = \frac{1-\alpha}{p_2} > 0$ (normal good) $\frac{\partial x_1}{\partial p_1} = 0$ (ordinary good)

Recall that from Slutsky's Equation with $i = j$:

$$\frac{\partial x_j^*(p,m)}{\partial p_j} = \frac{\partial x_{H,j}^*(p, v(p,m))}{\partial p_j} - \frac{\partial x_j^*(p,m)}{\partial m} \cdot x_j^*(p,m).$$

Since $x_{H,j}^* = \frac{\partial e(p,v(p,m))}{\partial p_j}$, and the expenditure function is concave in prices, we obtain for the sub. effect:

$$\frac{\partial x_{H,j}^*(p,v(p,m))}{\partial p_j} = \frac{\partial^2 e(p,v(p,m))}{\partial p_j^2} \leq 0 \quad \checkmark \text{ concavity}$$

Since demand $x_j^*(p,m) \geq 0$, we have the implication:

$$\frac{\partial x_j^*(p,m)}{\partial m} \geq 0 \Rightarrow \frac{\partial x_j^*(p,m)}{\partial p_j} \leq 0$$

That means a normal good is always an ordinary good.

Contraposition:

$$\frac{\partial x_j^*(p,m)}{\partial p_j} > 0 \Rightarrow \frac{\partial x_j^*(p,m)}{\partial m} < 0$$

That means a Giffen good is always an inferior good.

These considerations lead to a list of three necessary conditions for a good to be a Giffen good:

1) It must be an inferior good.

2) The sub. effect is relatively small.

3) A substantial part of income is spend on the respective good, but not all of it.

Part 2 – Markets and Competition

Markets – Demand, Supply, and Equilibrium

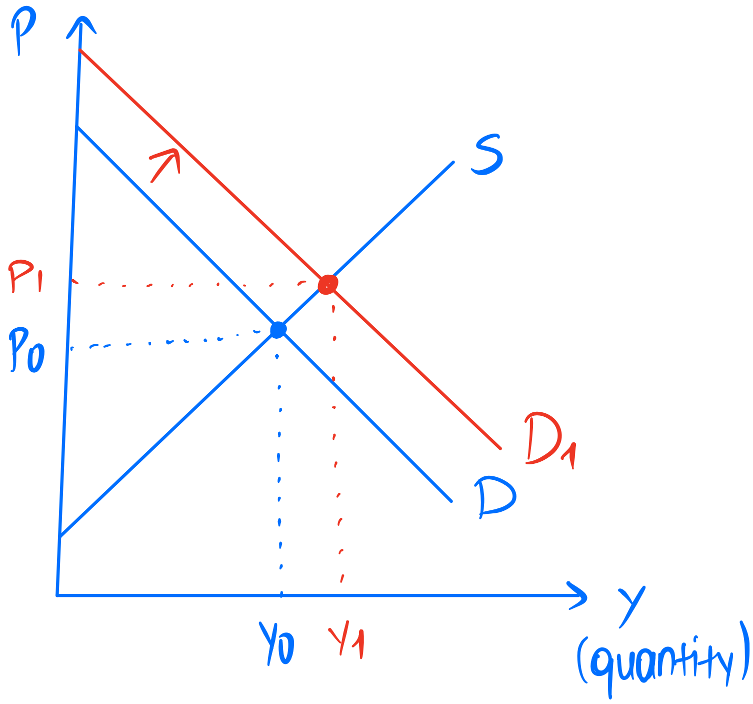
We define a market for a good or service to simply be the union of the individuals and firms that operate on both the supply and demand sides of a potential transaction. There are many different types of market that we should be aware of, some of which form rich areas of study themselves. We will continue to focus on markets for goods and services, however notable other markets include:

- Stock market
- Labour market
- Capital market

The intentions and wishes of each side of a market are, of course, specified through the demand and supply curves, and we recall from the start of the course that (for a competitive market) the prices at which goods are sold is settled through the price mechanism.

Recall that the price at which supply equals demand is referred to as the **equilibrium price**. Suppose that a market is settled at an equilibrium price p_0:

- If the demand for a good changes for some reason, then the demand curve will shift; demand and supply will no longer be equal at p_0.
- An increase in demand, or a decrease in supply, will lead to an ***excess in demand***.
- A decrease in demand, or an increase in supply, will lead to an ***excess in supply.***
- These excesses invoke the price mechanism, changing the equilibrium price. The speed at which this happens will vary between markets.



Excesses in demand or supply can be measured as long as we can express supply and demand in terms of the good's price.

- For individual consumers, demand is measurable in terms of Marshallian demand (under a utility maximisation assumption).

- For an individual firm, we have the (short-run and long-run) supply curve (under profit maximisation assumption).

We want the **market demand** and **market supply** (also referred to as the industry demand and industry supply)

Suppose a market for a single good contains I consumers and J firms. Further, suppose that consumer i has demand given by

$$x_i^*(p, m_i), \quad i \in \{1, \dots, I\}$$

and that the supply curve for firm j is specified by

$$y_j^*(p), \quad j \in \{1, \dots, J\}.$$

The market demand for the good is then defined as

$$X^*(p, m_1, \dots, m_I) = \sum_{i=1}^{I} x_i^*(p, m_i)$$

and the corresponding market supply is defined as

$$Y^*(p) = \sum_{j=1}^{J} y_j^*(p)$$

Example:

Suppose that the market for bananas contains 1000 utility-maximising consumers with demand functions

$$x_i^*(p, m_i) = m_i \frac{p_B}{p_B^2 + p_A^2}, \quad i = 1, \dots, 1000$$

(A: apple, B: banana)

Further, suppose the banana market comprises two suppliers, with supply curves

$$y_j^*(p) = \frac{p_B}{2j}, \quad j = 1, 2$$

What is the equilibrium price for bananas?

$$X^*(p, m_1, \dots, m_{1000}) = \sum_{i=1}^{1000} x_i^*(p, m) = M \frac{p_B}{p_B^2 + p_A^2},$$

where $M = \sum_{i=1}^{1000} m_i$

$$y^*(p) = y_1^*(p) + y_2^*(p) = \frac{p_B}{2} + \frac{p_B}{4} = \frac{3p_B}{4}$$

$$x^*(p, m_i) = y^*(p) \iff$$

$$M \frac{\cancel{p_B}}{p_B^2 + p_A^2} = \frac{3\cancel{p_B}}{4} \iff 4M = 3p_B^2 + 3p_A^2$$

$$\iff p_B = \pm\sqrt{\frac{4M}{3} - p_A^2}$$

$$\boxed{p_B = \sqrt{\frac{4M}{3} - p_A^2}}$$

Consumers' and Producers' Surplus – Social Welfare

To analyse the consequences of a change in prices or income – or more generally, a change in policy – it is useful to have a measure of social welfare. We will see that a handy such measure is the sum of ***consumers'*** and ***producers' surplus.*** It also gives rise to another characterization of the equilibrium price and equilibrium quantity, maximising this social welfare measure.

We put ourselves into the general framework of utility maximising consumers and profit maximising firms where the utility and production functions satisfy our usual assumptions. Suppose we have J firms with cost functions $c_j^*(\cdot)$, $j \in \{1, \dots, J\}$, and I consumers with respective utility functions $u_i(\cdot)$, $i \in \{1, \dots, I\}$, and corresponding quantities (i.e. indirect utility function v_i, expenditure function e_i, Marshallian demand x_i^*, and Hicksian demand $x_{H,i}^*$ as well as profit-maximising output y_j^*)

Optimal goal:

Consider fixed income levels $m_1, \dots, m_I$ and a price change from $\underline{p}^{(1)} \in \mathbb{R}_{\geq 0}^n$ to $\underline{p}^{(2)} \in \mathbb{R}_{\geq 0}^n$. Suppose that the price change affects only one single product and that w.l.o.g. the product gets more expensive. To save notation, we will only explicitly denote the variable with a price change, suppressing all the other ones. So in the specific good, we will consider a price change from $p^{(1)} > 0$ to $p^{(2)} > 0$ where we assume without loss of generality that $p^{(1)} < p^{(2)}$.
We assume that producers are concerned about their change of profit. So we can measure the effect of the price change with the quantity

$$\sum_{j=1}^{J} \left(\pi_j^*(p^{(2)}) - \pi_j^*(p^{(1)}) \right) = \sum_{j=1}^{J} \int_{p^{(1)}}^{p^{(2)}} \frac{d\pi_j^*(p)}{dp}\, dp$$

Recall that $\pi_j^*(p) = p y_j^*(p) - c_j^*(y_j^*(p))$

$$\frac{d}{dp}\pi_j^*(p) = (\pi_j^*(p))' = y_j^*(p) + p y_j^{*\prime}(p) - c_j^{*\prime}(y_j^*(p)) \cdot y_j^{*\prime}(p)$$

$$= y_j^*(p) + y_j^{*\prime}(p)\underbrace{\left(p - c_j^{*\prime}(y_j^*(p))\right)}_{=0}$$

$$= y_j^*(p)$$

$$\dots \sum_{j=1}^{J} \int_{p^{(1)}}^{p^{(2)}} y_j^*(p)\,dp = \int_{p^{(1)}}^{p^{(2)}} \sum_{j=1}^{J} y_j^*(p)\,dp$$

$$= \int_{p^{(1)}}^{p^{(2)}} Y^*(p)\,dp$$

Consequently, we introduce the ***producers' surplus at price*** $\hat{p}$ as one part of the measure for social welfare measure:

$$PS(\hat{p}) = \int_0^{\hat{p}} Y^*(p)\,dp.$$

The consumer side is a bit trickier. Following the utility maximisation rationale of the lecture, each individual consumer cares about the difference in their individual indirect utility, that is

$$v_i(p^{(2)}, m_i) - v_i(p^{(1)}, m_i)$$

However, this approach is problematic since

- Cannot aggregate ordinal utilities
- we would like to compare the effect to consumers with effect to producers

$\rightsquigarrow$ we need a monetary measure

A natural possibility is to consider the difference of the individual expenditure functions, keeping the initial (indirect) utility fixed. This quantity is known as ***compensating variation***

$$CV_i(p^{(1)}, p^{(2)}, m_i) = e_i(p^{(2)}, v_i(p^{(1)}, m_i)) - \underbrace{e_i(p^{(1)}, v_i(p^{(1)}, m_i))}_{=m_i}$$

$$= \int_{p^{(1)}}^{p^{(2)}} \frac{d}{dp} e_i(p, v_i(p^{(1)}, m_i))\,dp$$

$$= \int_{p^{(1)}}^{p^{(2)}} x^*_{H,i}(p, v_i(p^{(1)}, m_i))\,dp \quad \text{(Shephard's Lemma)}$$

<u>Problems</u>

1) Hicksian not observable

2) X^* defined as the sum of individual x_i^* (not $x^*_{H,i}$)

3) $CV_i(p^{(1)}, p^{(2)}, m_i) \neq -CV_i(p^{(2)}, p^{(1)}, m_i)$

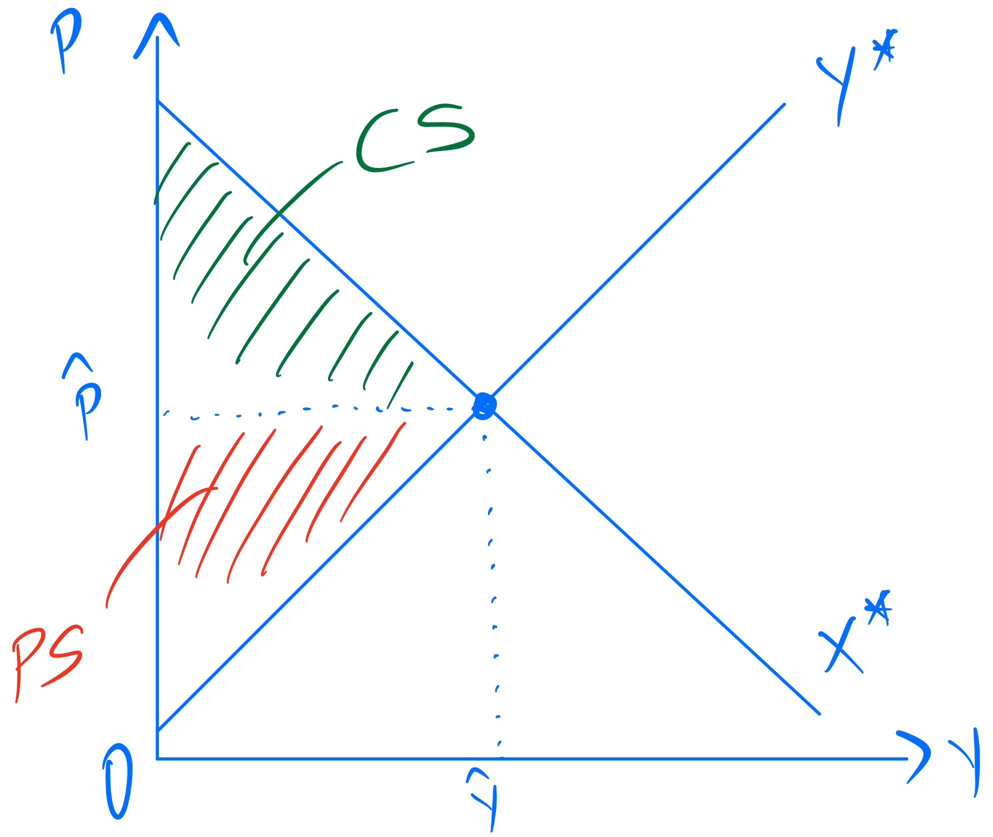
To this end consider $\boxed{\int_{p^{(1)}}^{p^{(2)}} x_i^*(p, m_i)\,dp}$

Assuming a good is normal, and using Slutsky's equation:

$$-CV_i(p^{(2)}, p^{(1)}, m) \leq \int_{p^{(1)}}^{p^{(2)}} x_i^*(p, m_i)\,dp \leq CV_i(p^{(1)}, p^{(2)}, m)$$

we define the consumer's surplus at price $\hat{p}$:

$$CS(\hat{p}) = \int_{\hat{p}}^{\infty} \sum_{i=1}^{I} x_i^*(p, m_i)\,dp = \int_{\hat{p}}^{\infty} X^*(p, m_1, \dots, m_I)\,dp$$



$SW = CS + PS$

social welfare

Finally, the sum of consumers' surplus and producers' surplus, the ***community surplus***, can be considered as a measure of social welfare.

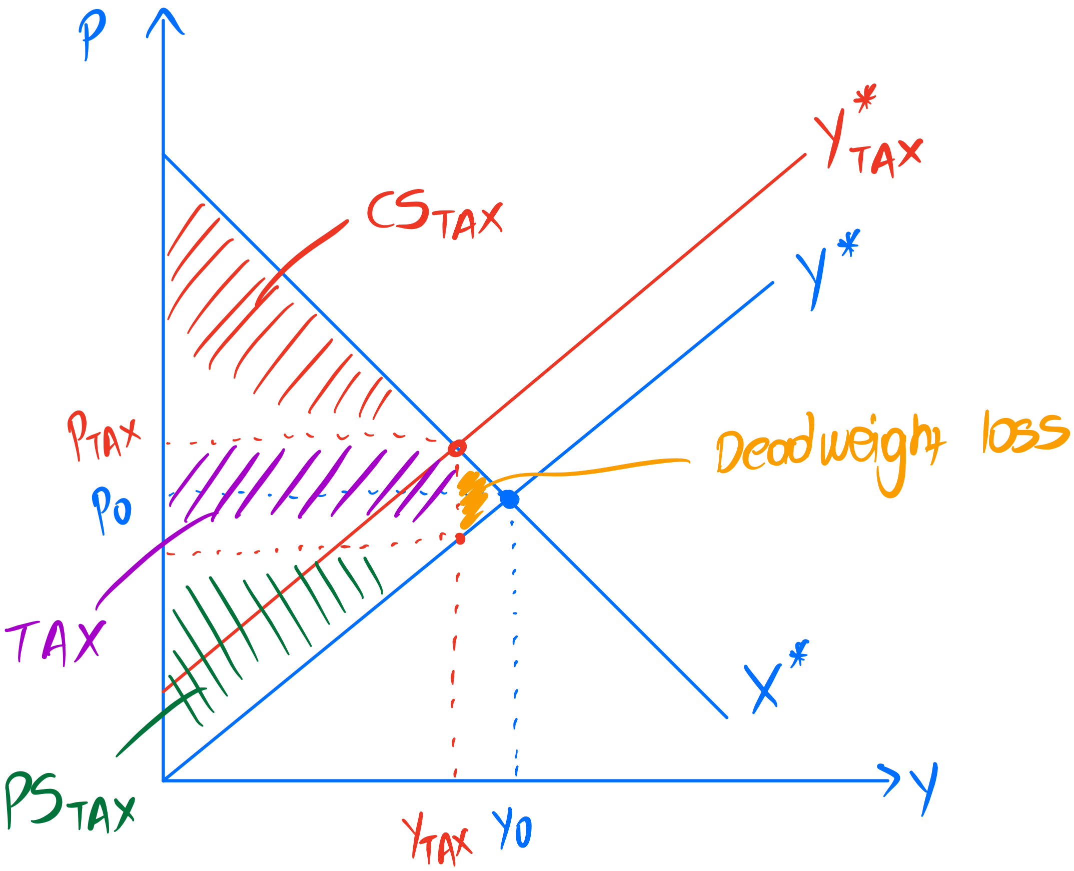
Changes in the market demand or industry supply of a good will lead to a change in its equilibrium price. Taxes and subsidies are an interesting example of factors that lead to such a change.

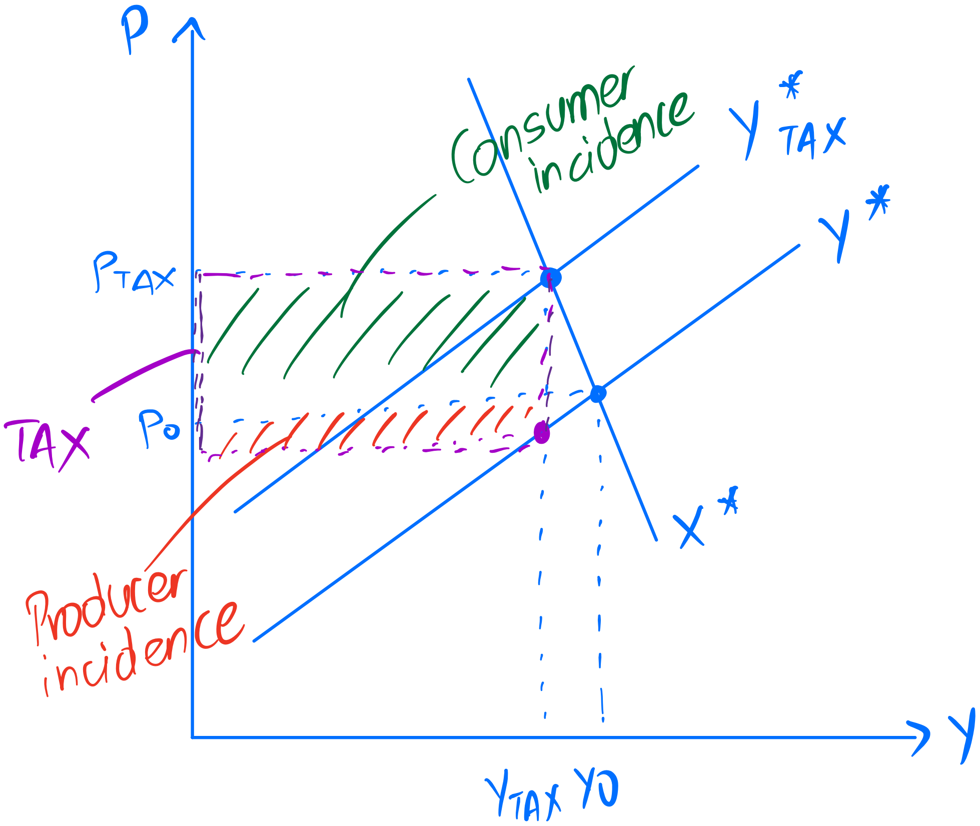
! speed

Indirect Taxes and Equilibrium:

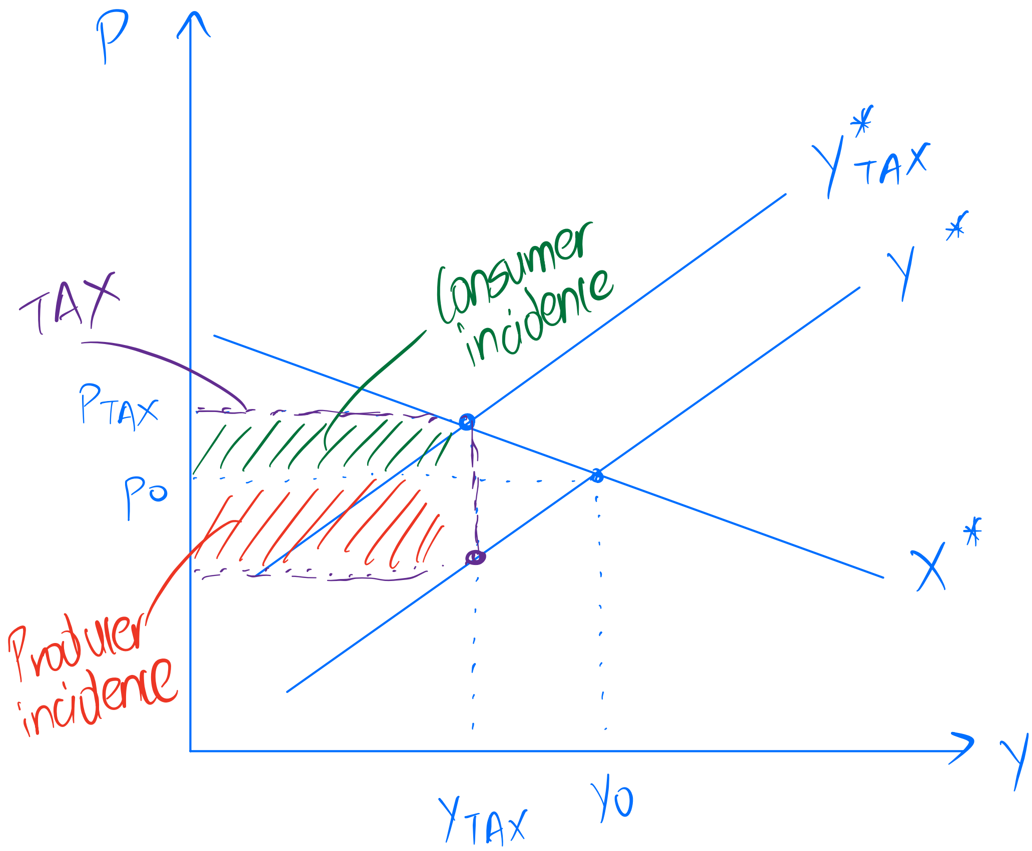
An indirect tax is one that can be passed on to another party. In the context of providing goods and services, an indirect tax on producers is one that is passed on to consumers. In general, a tax that is dependent on the quantity of good being produced can be treated as an indirect tax.

How much of an indirect tax is passed on to consumers?



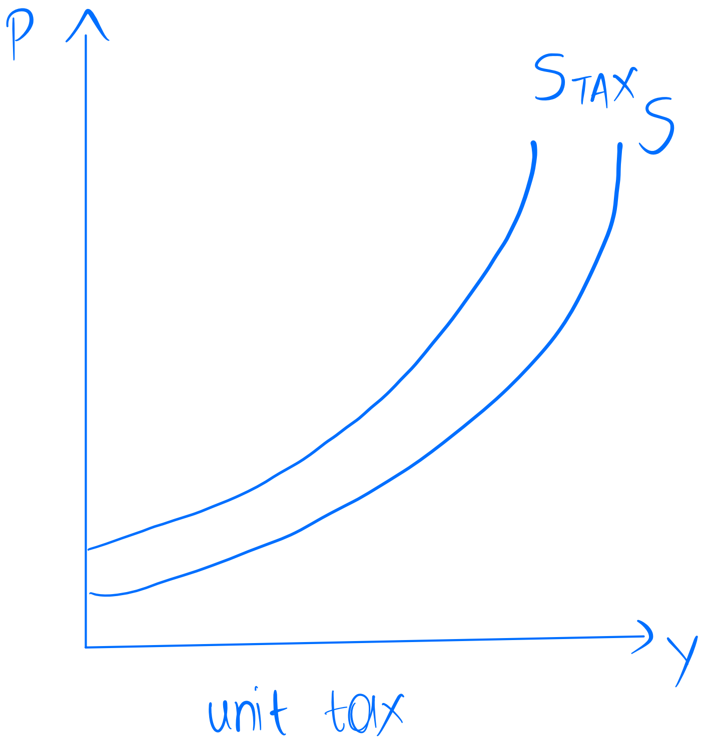


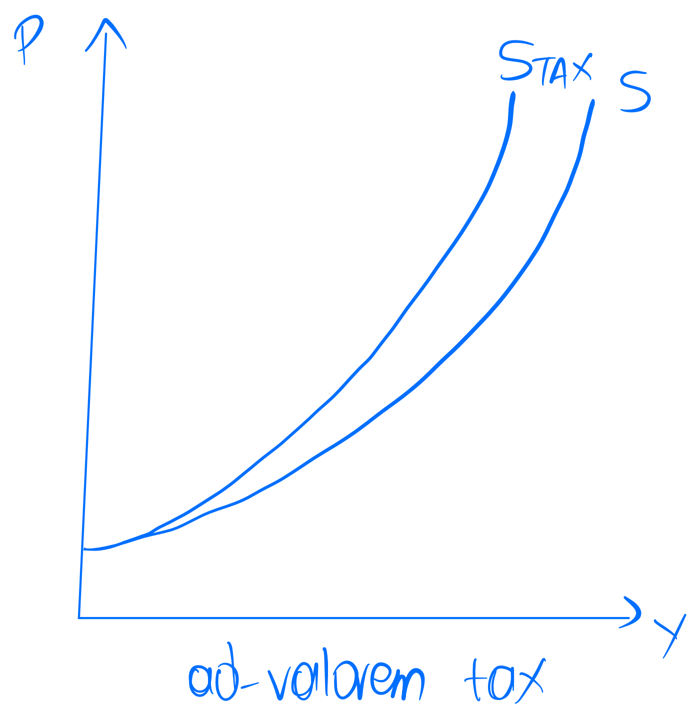
Price elasticity of supply exceeds price elasticity of demand



Price elasticity of demand exceeds price elasticity of supply.

Indirect taxes on the production of goods can be imposed in one of two ways:
- the tax may be a fixed amount per unit sold → unit tax or specifix tax, $p' = p + t$
- or it may be a percentage of the good's price → ad-valorem tax, $p' = (1+t)p$





In any case, imposing taxes reduces the community surplus. The difference between the original community surplus and the new community surplus plus the tax revenue is called ***deadweight loss***. However, if taxes are present, it is crucial that we include the government into the consideration and computation of the community surplus. That is, in the presence of taxes, the community surplus is the sum of the producers' surplus, the consumers' surplus and the tax revenue. (SW = PS + CS + Tax)

A deadweight loss occurs when the market price/quantity deviates from the equilibrium price/quantity. We have seen that taxes can cause a deadweight loss.

→ raise money for public services etc.
→ account for negative externalities, e.g. smoking, drinking

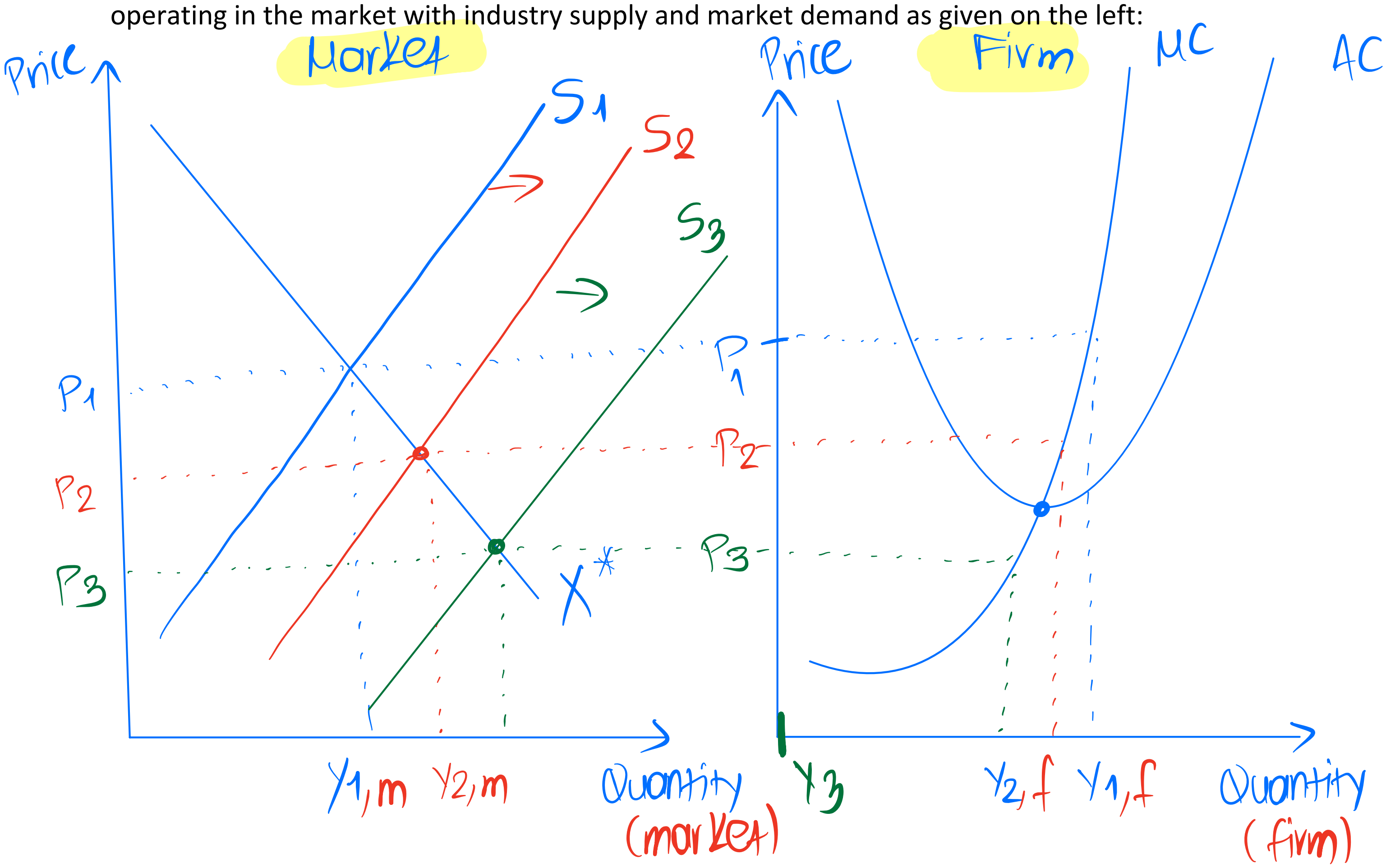
On the other hand, some activities are deemed to have positive externalities (education, culture, …), which functions as justifications for subsidies.

Subsidies can also cause a deadweight loss. Also in the presence of subsidies, one must include the government into the consideration and computation. That is, the community surplus is the sum of producers' surplus and consumers' surplus *minus* the total size of the subsidy. ($CW = PS + CS - Sub$)

Moreover, maximal or minimal prices as well as quantities can cause a deadweight loss.

Abnormal Profits, Long-run Equilibrium and Productive Efficiency

Consider a competitive firm and suppose it has costs given below, on the right, whilst operating in the market with industry supply and market demand as given on the left:



The firm's individual supply curve is obtained under the assumption that the firm is profit-maximising: by construction, we have that at any point on the firm's (positive) supply curve, their marginal revenue will equal their marginal cost.

Suppose the industry supply is given by S_1 in the short-run.

What will happen as we move into the long run?

Other firms want to enter the market.

↓

This will drive supply up (increase quantity for fixed price)

↓

until

marginal cost = average cost

Thus, the long-run equilibrium is the point at which no individual firm makes a profit.

But surely companies make long-run profits all the time?!

We are considering different types of profit here! This is where we need to distinguish between accounting costs and economic costs:

Accounting costs include all financial costs of production

- All paid costs
- Includes fixed and variable costs, e.g. wages, machinery, licence

Economic costs are accounting costs, plus ***opportunity costs***

Opp. costs:
- Foregone benefits by not choosing an alternative source of action.

Unit costs for each factor can be defined as economic costs
In the long run the opportunity costs should match the accounting profit.

Firms that exactly cover their economic costs have zero economic profit, but their accounting profit is equal to their opportunity costs they are said to be earning a normal profit.

Firms which more than cover their economic costs are said to be making an abnormal profit. This will encourage entry into the market by other firms.

Firms that make normal profits are said to be ***productively efficient*** – they produce at the minimum of the average cost curve, when taking opportunity costs into account.

Part 3 – Macroeconomics

In the following section, we will consider a number of fundamental concepts central to macroeconomic analysis.

Macroeconomy concerns country-wide economics and the economic variables that affect all firms and individuals to different extents. It also looks at the interaction of different (micro) markets and factors that influence the inter-operation of these markets. Such factors may include:

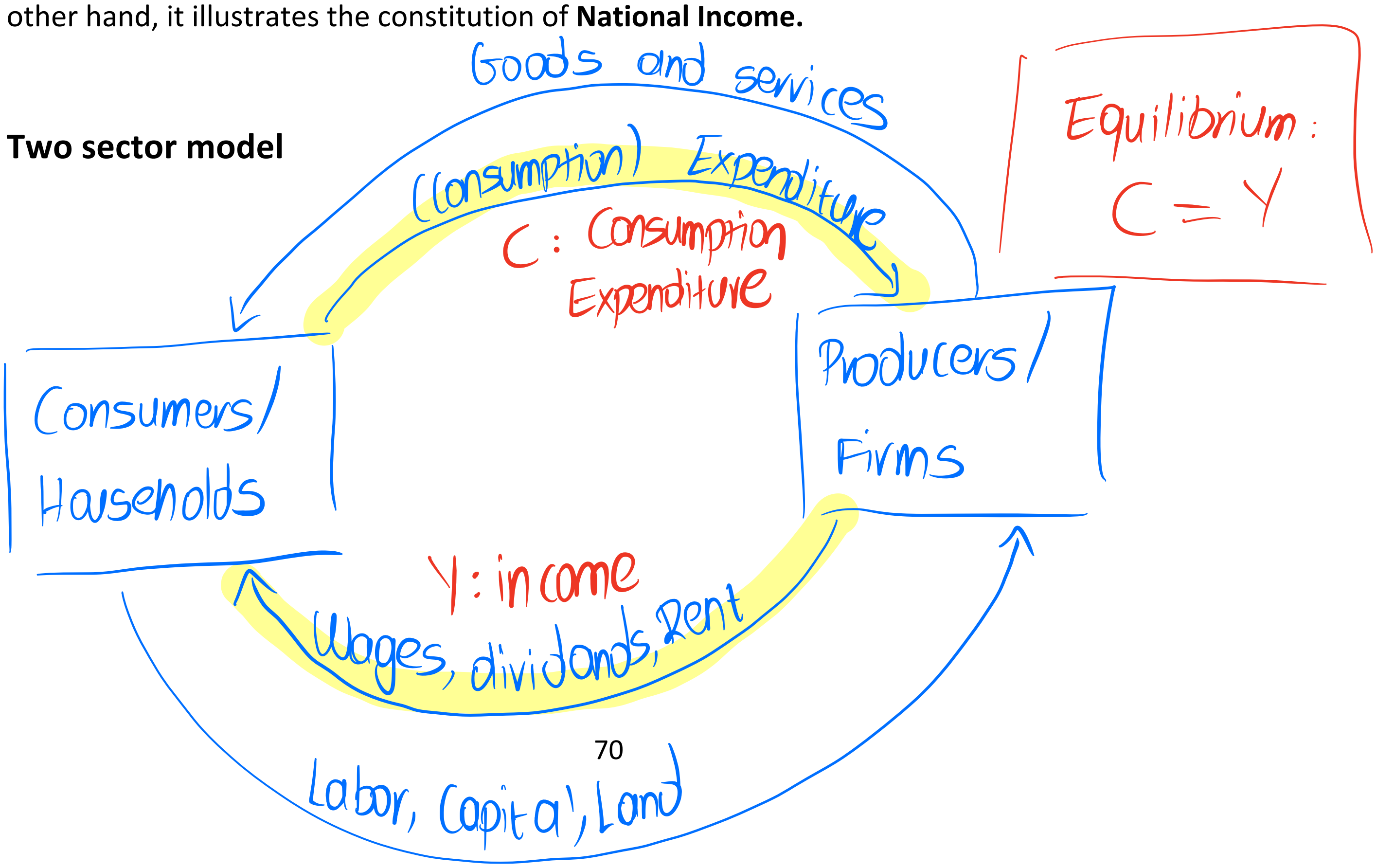
- Interest rates
- Exchange rates
- Inflation
- National income
- (Un)Employment
- Taxation

Although macroeconomics covers a much greater scale than microeconomics, there are some parallels. For example, an important concept in macroeconomic analysis is the idea of *aggregate* supply and *aggregate* demand. Note that these are separate concepts to the market supply and demand; here, we are talking about the entire economy, not just single markets!

The circular flow of income

The ***circular flow of income*** is a model that analyses how money, goods and services flow through the economy over a particular time horizon. It originates from the work of the French economist and physician François Quesnay (1694 – 1774) and is akin to the circulatory system of the body. Thus, it depicts the interdependence of the various economic agents. On the other hand, it illustrates the constitution of **National Income.**

Two sector model



We can actually see *two* circuits: One reflects actual physical goods (goods, services, factors of production/labor). The second is the *circuit of money* flowing into the opposite direction than the first one.

For the sake of simplicity, we will only denote one of the two circuits, which will be the one associated to money.

In order to quantify the flows, we introduce the following variables:

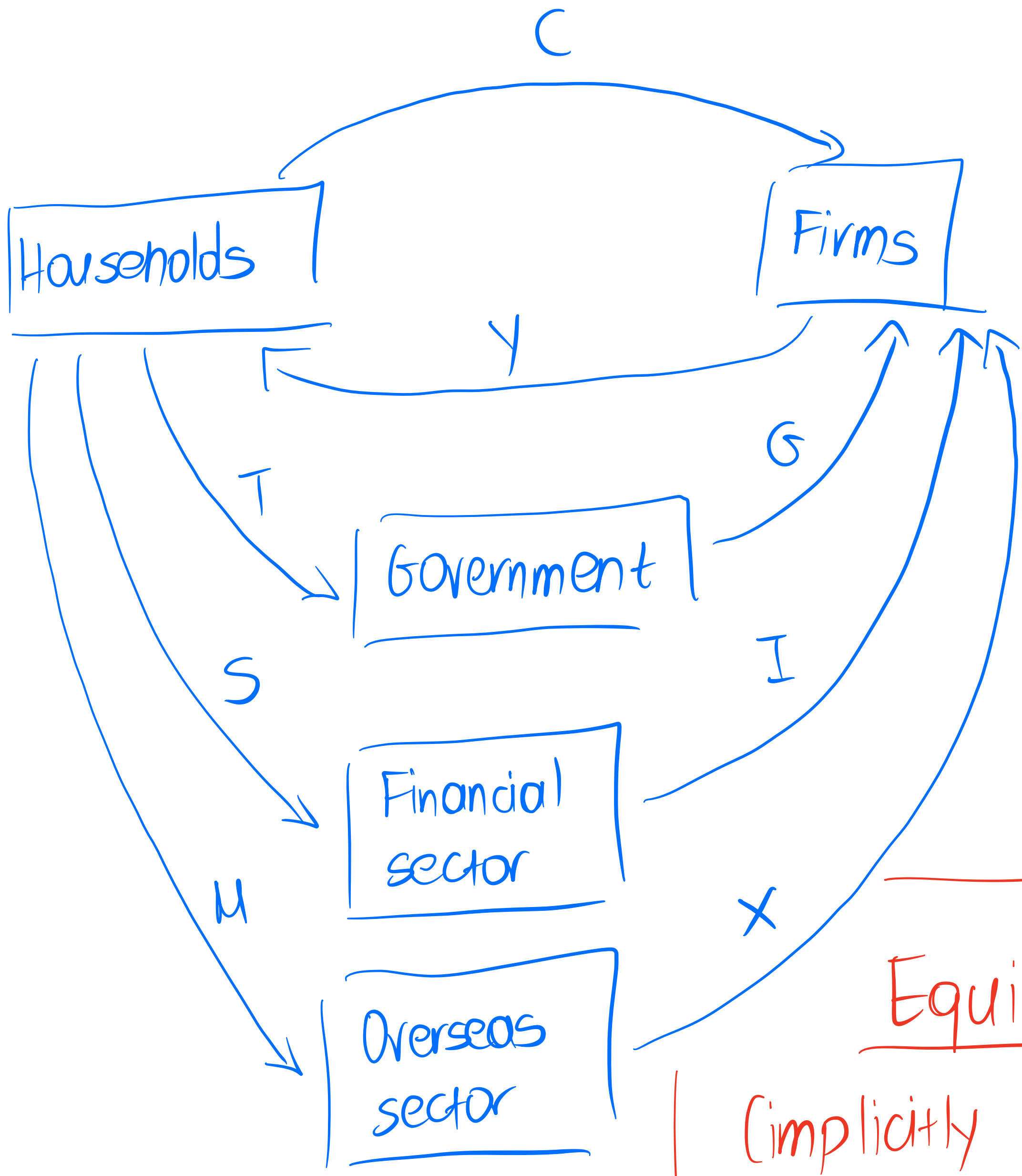
As the name suggests, the circular flow needs to be in **equilibrium** in the two sector model in the sense that

It has the interpretation that the households need to earn what they spend, but also spend what they earn. Another way to consider it is that demand (C) needs to be equal to supply (Y) or production. Now, one might wonder whether this model is too simplistic – do households always spend all their money in a certain period? And how can firms grow and increase their production capacities? We definitely need more agents in our model.

The five sector model

- Government sector
- Financial sector
- Overseas sector

5-sector model



Equilibrium:

(implicitly assume $C = Y$)

$$T + S + M = G + I + X$$

a) Leakages

Of course, not all of a household's income will be spent on domestic goods and services.

S : Savings
T : Taxes
M: Import spending

…these are all called 'withdrawals' or 'leakages' from the economy; they do not feed back into demand for domestic output

b) Injections

Additionally there will be 'injections' into the economy in the form of…

I : Investment spending
G: Government spending
X: Export spending

Injections and leakages such as these will obviously have an impact on the demand for all goods and services within the economy: leakages will reduce the aggregate demand and injections will boost it. Indeed, when the injections compensate for the leakages, then aggregate demand will equal aggregate supply, and the economy will be in equilibrium:

This is when: $I + G + X = S + T + M$

Indeed, we can be more specific and define ***aggregate demand*** as follows:

$$AD = \underbrace{I + G}_{\text{injections}} + \underbrace{C + X - M}_{\text{Domestic net consumption}}$$

Our definition of aggregate demand above requires careful consideration of various sources of demand for the final goods and services being provided to the economy by firms. In contrast, we can straightforwardly define ***aggregate supply*** to be the total value of all $\boxed{\text{final}}$ goods and services provided by firms to the economy.

final vs intermediate

e.g. car steel

Recall that equilibrium in the economy occurs when $I + G + X = S + T + M$. These variables can be paired according to the respective sectors:

- **Financial sector (I, S):** In order to make investments, firms obtain the required financing from financial institutions; these institutions are able to provide the financing due to the savings of consumers.

  I might not equal S – decisions to save/invest are made by different parties

- **Government sector (G, T)**: The government can both inject and withdraw from the economy, via spending and taxation, respectively.

  G might not equal T – governments might choose to run a budget ~~deficit~~ surplus $(T > G)$ or budget ~~surplus~~ deficit $(G > T)$.

- **Overseas sector (X, M)**: Demand for exports and demand for imports are also linked, but may not necessarily be equal.

  Trade surplus $(X > M)$, e.g. China

  Trade deficit $(M > X)$, e.g. US

  Trade balance = value of exported goods minus value of imported goods.

Considering the second pairing in particular, we see that governments can therefore influence the economy by forcing a mismatch in withdrawals and injections.

One can qualitatively discuss what happens if the economy deviates from the equilibrium. Suppose that, over a given time period, injections exceed withdrawals:

- There will be an excess in aggregate demand, motivating an increase in aggregate supply to move towards equilibrium. → economic growth!

- The resulting increase in aggregate supply may cause firms to increase their labour supply, leading to a fall in unemployment.

- An increase in demand will also increase prices → inflation?

- Excess in demand will increase imports as consumers buy elsewhere; exports will decrease due to rising prices.

positive trade balance → TS

negative trade balance → TD

Gross Domestic Product (GDP)

How do we exactly measure the overall production Y of an economy? We also need this measure to talk about deviations in production, which can be recession or growth.
So we introduce one of the most famous notions of macroeconomics.

Def: The Gross Domestic Product (GDP) measures the nominal gross value of all goods and services produced in a certain country in a certain time period.

Comments:

- The GDP is a monetary / nominal value, not a real value.
- It measures only the gross value which is *added* in the course of production. → Avoiding double counting.
- How to measure services provided by the state (e.g. administration, education, security, defense)? → costs
- The confining quantities are time, but also area – that's why it's called *domestic.* A (historic) alternative is the ***Gross National Product (GNP)***. The difference to GDP is that GNP measures the gross output of all *citizens* of a certain nationality irrespective of their residence.
  It was used to measure the overall production foremost until the first half of the 20th century.

Different ways to calculate the GDP

We start with an example: Consider two firms.

F1 (Steel producer):

- Revenue: £100
- Wages: £50
- Capital: £30
- Profit: £20

F2 (Car producer):

- Revenue: £210
- Steel: £100
- Wages: £70
- Profit: £40

1) **Production Approach:** Calculate the *gross value added* in the *domestic* production. This is gross value of output minus intermediate consumption. In our example, this is

$$GDP = \underbrace{(£100 + £210)}_{\text{gross value of output}} - \underbrace{£100}_{\text{intermediate consumption}} = £210$$

This approach reflects the very definition of GDP probably best. Note that the GDP avoids double counting. A good intuitive justification for this is to imagine that F1 and F2 merged. Then we would not see the intermediate consumption (it would not be reported to the national statistics office). In macroeconomics, one considers the entire supply side as one firm such that this perspective makes sense.

How would GDP change if F1 were an overseas steel supplier?

$$GDP = £210 - £100 = £110$$

2) **Expenditure approach:** This approach takes the angle that everything that was produced has to be bought. We can use the circular flow diagram:

$$Y = C + I + G + X - M$$

In our example, we have

$$C = £210, \quad I = G = X = M = 0$$

$$GDP = £210$$

Again, if F1 were overseas, we had

$$C = £210, \quad I = G = X = 0, \quad M = £100$$

$$GDP = £210 - £100 = £110$$

3) **Income approach:** Somebody has to earn the value that has been created. This is usually income from labour, capital, and taxes, which was generated domestically. In our example, that is

$$GDP = \underbrace{£50 + £70}_{\text{labour}} + \underbrace{£30 + £20 + £40}_{\text{Rents, interests, dividents}} = £210$$

Again, for the alternative scenario with F1 being abroad, we get

$$GDP = £70 + £40 = £110$$

Criticism concerning GDP as an overall welfare measure

1. Since GDP is a nominal value, price changes (due to inflation) can cause an increase in GDP (but they do not affect production in real terms – at least primarily).

2. Many services are ignored, e.g., child-rearing, care for elderly people, working in an honorary capacity (in societies, congregations, sports, …)

3. Externalities are often ignored, e.g., adverse effects to the environment

4. Depreciation is often ignored, e.g., destruction of infrastructure by (natural) disasters such as storms, floods, but also war. Depreciation can also come from natural sources.

5. It ignores the benefits of leisure.

GDP is often used as a proxy or an indicator of the overall welfare of a society. To some extent, this is certainly justified. However, what can happen if decision-makers in politics, business, and society are mixing up the notion of an indicator with overall welfare itself?

They might try to increase GDP without actually improving the overall welfare of society (or maybe even deteriorating welfare).

1) Inflation increases GDP without increasing the real output.

2) Tendency to commercialise those services such as child-rearing, care for elderly people, etc.

3) One might come up with lower standards in environmental regulation, thus increasing GDP.

4) There might be an incentive to cause depreciation in order to re-build infrastructure. Also, in business there might be an incentive to produce products with a high deterioration rate.

5) People might be pushed into (dependent) work despite their preferences.

Can you think of possibilities how solve these problems in the practical usage of GDP as a proxy for overall welfare of a society?

Allocation of income – connections to social welfare

There is an ongoing debate (over the last centuries) how income should be distributed. Before attacking this *normative* question, let us first turn to the *descriptive* side of it. How can the distribution of income been measured?

Certainly, this is a statistical question. The most informative measure would be to report each individual's income (which amounts to reporting the empirical distribution function of income). However, this is too complex to report. So we are actually looking for some number that summarises the distribution of income.

Suggestions:

- Mean
- Median
- Variance / st. deviation
- Range
- IQR
- Several quantiles

What is often of interest is a ***relative measure for dispersion*** in order to measure how equally income is distributed. There are several ways how to do this.

e.g. coefficient of variation

A measure that is most commonly used for this purpose is the ***Gini coefficient***. For a population of n persons with income $y_1, \dots, y_n$ it is defined as

$$G(y_1, \dots, y_n) = \frac{\sum_{i,j=1}^{n} |y_i - y_j|}{2n \sum_{i=1}^{n} y_i}$$

Indeed, one can show that the Gini coefficient is

- Positively homogeneous of degree 0 in $y_1, \dots, y_n$;
- Always between 0 and 1
- 0 if and only if there is complete equality. That is, if and only if $y_1 = \cdots = y_n$;
- For fixed n it is maximal if and only if there is complete inequality. That is, if and only if there is some $k \in \{1, \dots, n\}$ such that $y_k = \sum_{j=1}^{n} y_j$. Indeed, then
$$G = \frac{n-1}{n}.$$

Normative positions

Besides merely describing income and wealth allocation, there is an ongoing debate between different *normative* positions. Recall that two of the main drawbacks in this discussion is that utilities are not directly observable. And moreover, since we are working with ordinary utilities, there is no way how to meaningfully aggregate them.

Here is an overview of some positions, including possible criticisms.

- Equal income distribution: This position asserts that everybody should have the same income which is akin to communistic theory.
  What is problematic is that there are only week (monetary) incentives to contribute to economic growth (at least if we assume the *Homo Economicus*).

- Minimax approach: Suppose you were born into a society with uncertainty in which class you will be born. The most risk averse approach would then be to minimise the maximally adverse outcome (therefore the name). Equivalently, one would try to maximise the minimal utility in the population.
  Criticism: One needs to compare different utility functions.

- **Pareto efficiency:**
  A Pareto improvement is a change that makes at least one person better off, without making any other person worse of. An allocation is Pareto efficient if no Pareto improvement is possible.
  Even though this is a widely agreed criterion for income allocation, it is also a rather week notion of efficiency. E.g. if utilities are strictly increasing in income, also an allocation is Pareto efficient where one person owns the entire income and the rest does not have anything at all.

- An alternative perspective is that one should not care about the *outcome* of the income allocation, but only about the underlying mechanism. If the allocation mechanism is fair (if it works according to fair rules and laws), then any resulting allocation is fair.
  This position is probably most akin to a purely capitalistic approach.

In the entire discussion, it is crucial to make a precise distinction between relative and absolute allocation of income. Bear in mind that a third of a very large cake might still be better than half of a rather small cake.

However, some studies show that people often rather care about their relative income…