Mathematics of Business and Economics Concise Notes

MATH60013

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Content from prior years assumed to be known.

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Contents

Part I
Microeconomics

1 Supply and Demand - an introduction

Definition 1.1. ***Supply*** *refers to the quantity of a good that vendors are willing and able to sell at a given price.*

Definition 1.2. ***Demand*** *refers to the quantity of a good that consumers are willing and able to buy at a given price.*

Theorem 1.3. ***The Law of Demand*** *states that the quantity demanded of a good is inversely related to its price. That is, as the price of a good increases, the quantity demanded of that good decreases.*

Theorem 1.4. ***The Law of Supply*** *states that the quantity supplied of a good is directly related to its price. That is, as the price of a good increases, the quantity supplied of that good increases.*

1.1 Supply and Demand Curves

Definition 1.5. ***A supply curve*** *is a graph of the quantity supplied of a good at different prices.*

Definition 1.6. ***(Price elasticity of supply/demand)*** *is the percentage change in quantity supplied/demanded divided by the percentage change in price.*

$$\begin{aligned}\epsilon_D &= \frac{\textit{proportional change in quantity demanded}}{\textit{proportional change in price}} \\ &= \frac{\partial D(p)}{\partial p}\frac{p}{D(p)}\end{aligned}$$

We have generally that $\epsilon_D < 0$ by the Law of demand, as demand is decreasing in p.

$$|\epsilon_D(p)| = \begin{cases} > 1, & \textit{elastic demand;} \\ < 1, & \textit{inelastic ;} \\ = 1, & \textit{unit elastic .} \end{cases}$$

Determinants of elasticity include:

- *The availability of substitutes*
- *The time period over which the price change occurs*
- *The proportion of income spent on the good*
- *The proportion of the good that is a necessity*

Define the ***price elasticity of supply*** *ϵ_S similarly.*
Can also consider the following:

- ***income elasticity of demand*** *ϵ_Y - the percentage change in quantity demanded divided by the percentage change in income.*
- ***cross-price elasticity of demand*** *ϵ_X - the percentage change in quantity demanded of good X divided by the percentage change in the price of good Y.*

1.2 Demand, Price and Revenue

Definition 1.7. ***Revenue*** *generated by a given good is the price of that good multiplied by the quantity demanded of that good.*

$$R(p) = Q(p) \cdot p = D(p) \cdot p$$

Theorem 1.8. *An increase in price will lead to a increase in revenue iff the demand for the good is inelastic When demand for a good is elastic, an increase in price will lead to a decrease in revenue, and vice versa.*

$$\begin{aligned}\frac{\partial R(p)}{\partial p} &= \frac{\partial D(p)}{\partial p} \cdot p + D(p) > 0 \iff \\ &= -\frac{\partial D(p)}{\partial p} \cdot p < D(p) \iff \\ &= -\frac{\partial D(p)}{\partial p} \cdot \frac{p}{D(p)} < 1 \\ &= -\epsilon_D < 1\end{aligned}$$

1.3 Theory of the Firm

1.3.1 Production Functions, Cost, Revnue and Profit

Definition 1.9. ***Factors of production*** *are the inputs used to produce a good. These fall into 4 categories:*

- *Land*
- *Labour*
- *Capital*
- *Raw materials*

Capital *is the those inputs to production that may be consumed now, but will deliver greater overall value ot the firm if consumption is delayed.*
Capitol goods *are those inputs to production that are themselves produced goods which are durable and used to produce other goods. i.e machinery, equipment*
Capitol finance *refers to the financial assets of a firm that are used to generate wealth - differs from 'money' in general in that it is not used to purchase consumable goods and services e.g shares, bonds, patents*

Definition 1.10. ***Production set*** *- the set of all inputs and outputs that satisfy a given restraint.*

$$\textit{Vector of input factor quantities} = \vec{x} \in \mathbb{R}^n_{\geq 0}$$
$$\textit{Vector of output factor quantities} = \vec{y} \in \mathbb{R}^m_{\geq 0}$$
$$\mathcal{P} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid \; s.t \; y \leq f(x)\} \quad where \quad f(x) \in \mathbb{R}^m_{\geq 0} \textit{ a production function}$$

The production function f *also known as the technology of the firm, prescribes the maximum level of output* $\vec{y}$ *for a given level of input* $\vec{x}$.

Definition 1.11. ***Isoquant*** *- a curve that connects all points in the production set that have the same output level.*

$$\textit{For given } \vec{y} \in \mathbb{R}^m_{\geq 0}, \textit{ the isoquant is the set of all } \vec{x} \in \mathbb{R}^n_{\geq 0} \textit{ such that } y = f(x)$$

Definition 1.12. *When* $y \in \mathbb{R}_{\geq 0}$, *the* ***input requirement set*** *is the set of all vectors* $\vec{x}$ *that produce at least* y *that is* $f^{-1}([y, \infty))$

Leontief Technology
Perfect substitutes
Cobb-Douglas Technology

1.3.2 Properties of production functions/ input requirement sets

- Monoticity - If some of the inputs is increased, the maximum will not decrease.

$$\vec{x} \leq \vec{x}^* \Rightarrow f(\vec{x}) \leq f(\vec{x}^*)$$

- Convexity

$$x$$

1.3.3 The Marginal Product

Definition 1.13. ***Marginal product of factor*** i *defined as*

$$\mu P_i(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_i}, \quad i = 1, 2$$

Definition 1.14. ***Output elasticity*** *with respect to factor i is defined as*

$$\epsilon_i(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_i} \cdot \frac{x_i}{f(x_1, x_2)}$$

1.3.4 Substitution

Definition 1.15. ***Marginal rate of technical substitution (MRTS)*** *is defined as*

$$MRTS(x_1, x_2) = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)}$$

Theorem 1.16. *(Law of diminishing marginal productivity)*
Increasing the quantity of a factor of production will increase the output of the firm, but at a decreasing rate.

1.3.5 Returns to Scale

Now consider the effect of increasing the quantity of all input factors of production by the same constant. Consider behaviour of function

$$\mathbb{R}_{\geq 0} \ni t \mapsto f(t\vec{x})$$

We have the following scenarios:

- **Constant returns to scale** - if $f(t\vec{x}) = tf(\vec{x})$
- **Increasing returns to scale** - if $f(t\vec{x}) > tf(\vec{x})$
- **Decreasing returns to scale** - if $f(t\vec{x}) < tf(\vec{x})$

Formalising the above quantitatively
Define the following function

$$h_x(t) = f(t\vec{x})$$

If f differentiable function we define its elasticity of scale at $\vec{x} \in \mathbb{R}^n_{\geq 0}$ as

$$\begin{aligned} e(\vec{x}) &= \frac{dh(t)}{dt} \cdot \frac{t}{h(t)} \mid_{t=1} \\ &= \frac{h'(1)}{h(1)} \\ &= \frac{df(t\vec{x})}{dt} \mid_{t=1} \cdot \frac{1}{f(x)} \\ &= \frac{\langle \nabla f(\vec{x}), \vec{x} \rangle}{f(x)} \end{aligned}$$

With the following cases:

- $e(\vec{x}) \geq 1, \ \forall \vec{x} \in \mathbb{R}^n_{\geq 0} \iff f(t\vec{x}) \geq tf(\vec{x}) \ \forall t \geq 1 \ \forall \vec{x} \in \mathbb{R}^n_{\geq 0}$
- $e(\vec{x}) \leq 1, \ \forall \vec{x} \in \mathbb{R}^n_{\geq 0} \iff f(t\vec{x}) \leq tf(\vec{x}) \ \forall t \geq 1 \ \forall \vec{x} \in \mathbb{R}^n_{\geq 0}$
- $e(\vec{x}) = 1, \ \forall \vec{x} \in \mathbb{R}^n_{\geq 0} \iff f(t\vec{x}) = tf(\vec{x}) \ \forall t > 0 \ \forall \vec{x} \in \mathbb{R}^n_{\geq 0}$

1.3.6 Additional potential properties of the production function

- **Homogeneity** $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is positively homogeneous of degree $k \in \mathbb{R}$ if $f(\lambda\vec{x}) = \lambda^k f(\vec{x}) \ \forall t > 0 \ \forall \vec{x} \in \mathbb{R}^n_{\geq 0}$
  This has obvious links to the returns to scale
  If f is positively homogeneous function of degree $k \in \mathbb{R}$
  - it has IRTS if $k > 1$
  - it has CRTS if $k = 1$
  - it has DRTS if $k < 1$
- **Homotheticity** $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is homothetic if there is a positively homogeneous function $h : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ and a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that

  $$f(\vec{x}) = g(h(\vec{x})) \quad \forall \vec{x} \in \mathbb{R}^n_{\geq 0}$$

  Homogeneous and homothetic production functions are useful modelling scenarios as they prescribe isoquants that vary simply for differing levels of output.

1.3.7 Profit and profit maximisation

Definition 1.17. *(Profit function)*

$$\begin{aligned}\pi(\vec{x}, p, \vec{w}) &= pf(\vec{x})^T - \vec{w}\vec{x}^T \\ &= pf(\vec{x}) - \vec{w}\vec{x}^T \quad \textit{(single output case)}\end{aligned}$$

$\vec{x}$ - the input factors, $\vec{w}$ - price to buy input, $\vec{p}$ - price to sell at

First order conditions for *π (Fundamental condition of profit maximisation)*

$$\forall i = 1, \ldots, n \quad \frac{\partial \pi}{\partial x_i} = 0 \iff \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i} p_j = w_i$$

- *Profit is maximised if marginal revenue is equal to marginal cost*
- *'The value of the marginal product of a factor of production is equal to the price of the product'*

For fixed prices and wages, the profit function is a function of the input factors only - can then be maximised by solving the first order conditions for x_i

$$f''(x^*) \leq 0 \ \textit{(necessary)} \quad \textit{and} \quad f'(x^*) < 0 \ \textit{(sufficient)} \quad \textit{(single-input case)}$$

For $\vec{x} \in \mathbb{R}^n_{\geq 0}, n > 1$ corresponding condition is that the Hessian of the profit function is negative semidefinite (necessary) or negative definite (sufficient)

Drawbacks

- Production function may not be differentiable
- Input variables may be discrete and not continuous
- Might have boundary solutions (e.g. $x_i = 0$) meaning that it optimal to not use a certain factor at all

- A best strategy might not exists e.g π might not be bounded, $f(x) = x$ and $w < p$
- The optimal strategy may exist but is not unique e.g $f(x) = x$ and $w = p$

Definition 1.18. ***Factor demand function***

$$\underline{x}^* : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to 2^{\mathbb{R}^{\ltimes}_{\geq \nvdash}}$$
$$\underline{x}^*(p, \vec{w}) = \arg\max_{\vec{x} \in \mathbb{R}^n_{\geq 0}} \pi(\vec{x}, p, \vec{w})$$

Gives the optimal specification of input quantities, given a price vector p and a wage vector $\vec{w}$

Output supply *as the function*

$$\underline{y}^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m_{\geq 0} \to \mathbb{R}^m_{\geq 0}$$
$$(\vec{p}, \vec{w}) \mapsto \underline{y}^*(\vec{p}, \vec{w}) = f(\underline{x}^*(\vec{p}, \vec{w}))$$

Profit Function

$$\pi^* : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}$$
$$(\vec{p}, \vec{w}) \mapsto \pi^*(\vec{p}, \vec{w}) = \pi(\underline{x}^*(\vec{p}, \vec{w}), \vec{p}, \vec{w}) = \max_{\vec{x} \in \mathbb{R}^n_{\geq 0}} \pi(\vec{x}, \vec{p}, \vec{w}) = \max_{\vec{x} \in \mathbb{R}^n_{\geq 0}} \left(\vec{p} f(\vec{x})^T - \vec{w}\vec{x}^T\right)$$

Properties:

- *Factor demand function is positively homogeneous of degree 0*

$$\underline{x}^*(t\vec{p}, t\vec{w}) = \underline{x}^*(\vec{p}, \vec{w}) \quad \forall t > 0 \quad \forall (\vec{p}, \vec{w}) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$$

- *Profit function is positively homogeneous of degree 1*
- *Profit function is non-decreasing in* $\vec{p}$ *and non-increasing in* $\vec{w}$
- *Profit function is convex*
- *Under regularity conditions, the profit function is continuous.*

Theorem 1.19. *(Hotelling's Lemma)*

Output supply and factor demand functions can be obtained directly from maximised profit function via partial differentiation w.r.t price vector

$$\pi^*(\vec{p}, \vec{w}) = \vec{p} f(\underline{x}^*(\vec{p}, \vec{w}))^T - \vec{w}\underline{x}^*(\vec{p}, \vec{w})^T$$
$$\frac{\partial \pi^*(\vec{p}, \vec{w})}{\partial p_j} = f_j(\underline{x}^*(\vec{p}, \vec{w}))$$
$$\frac{\partial \pi^*(\vec{p}, \vec{w})}{\partial w_i} = -x_i^*(\vec{p}, \vec{w})$$

Theorem 1.20. *(The LeChatelier principle)*

Long-run supply response to change in price is at least as large as the short-run supply response

$$\frac{\partial y_i^*(\vec{p}, \vec{w})}{\partial p_i} \geq 0$$

1.3.8 Weak Axiom of Profit Maximisation (WAPM)

Suppose observe net output vectors $z^t = (\underline{y}^t - \underline{x}^t) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$
And corresponding price vectors $\underline{r}^t = (\underline{p}^t, \underline{w}^t) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0}$ for a firm at discrete times $t = 1, 2, \ldots$

Assuming that the firm is profit maximising, then the following holds

$$\underline{r}^t(\underline{z}^t)^T \geq \underline{r}^t(\underline{z}^s)^T \quad \forall s, t = 1, \ldots, T$$
$$\Rightarrow \underline{r}^t(\underline{z}^t - \underline{z}^s)^T \geq 0, \quad \underline{r}^s(\underline{z}^t - \underline{z}^s)^T \geq 0$$
$$(\underline{r}^t - \underline{r}^s)(\underline{z}^t - \underline{z}^s)^T \geq 0 \quad \forall s, t = 1, \ldots, T$$

1.3.9 Cost Minimisation

Aim to answer 2 questions

- For fixed $\vec{w} \in \mathbb{R}^n_{\geq 0}$ what is the minimum total cost to the firm of producing level of output y
- What is the most profitable level of output?

Want to solve, given $\vec{w} \in \mathbb{R}^n_{\geq 0}$ and $\vec{x} \in \mathbb{R}^n_{\geq 0}$, the following problem

$$\arg\min_{\vec{x} \in f^{-1}(\{y\})} \vec{w}\vec{x}^T$$

Definition 1.21. *(Lagrangian)*

$$\mathcal{L}(\vec{x}, \lambda) = \vec{w}\vec{x}^T - \lambda(f(\vec{x}) - \vec{y})$$

$$\begin{aligned}
\frac{\partial \mathcal{L}(x, \lambda)}{\partial x_i} &= w_i - \lambda \partial_i f(x) = 0 \quad i = 1, \dots, n \\
\frac{\partial \mathcal{L}(x, \lambda)}{\partial \lambda} &= f(x) - y = 0 \\
&\Rightarrow \textit{ Must solve following } n+1 \textit{ unknowns} \\
\lambda \partial_i f(\vec{x}) &= w_i, \quad i = 1, \dots, n \\
f(\vec{x}) &= y
\end{aligned}$$

Definition 1.22. ***Conditional factor demand function***

$$\underline{x}^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \quad (\vec{w}, y) \mapsto \arg\min_{\vec{x} \in f^{-1}(\{y\})} \vec{w}\vec{x}^T$$

Minimum total cost to firm of producing level of output y given factor prices $\vec{w}$

$$c^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \quad (\vec{w}, y) \mapsto \min_{\vec{x} \in f^{-1}(\{y\})} \vec{w}\vec{x}^T = \vec{w}\underline{x}^*(\vec{w}, y)^T$$

First-order conditions restated as

$$MRTS(x_1^*, x_2^*) = -\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = -\frac{w_1}{w_2}$$

We have MRTS coincides with the ***economic rate of substitution***

Drawbacks of finding conditional factor demand function

- Production function may not be differentiable
- May have boundary solutions - meaning not optimal to use certain factor in production at all
- If production function f is continuous, surjective on $[0, \infty)$ and $w \gg 0$ there is always a cost minimizing strategy
- The optimal strategy may exist but may not be unique

1.3.10 Properties of the cost function

- Non-decreasing in $\vec{w}$
$$\vec{w}' \geq \vec{w} \implies c^*(\vec{w}', y) \geq c^*(\vec{w}, y)$$
- Homogeneous of degree 1 in $\vec{w}$
$$c^*(t\vec{w}, y) = tc^*(\vec{w}, y) \quad \forall t > 0$$

- Concave in $\vec{w}$
$$c^*(t\vec{w} + (1-t)\vec{w}', y) \geq tc^*(\vec{w}, y) + (1-t)c^*(\vec{w}', y) \quad \forall t \in [0,1]$$
- Continuous in $\vec{w}$ for $\vec{w} \gg 0$
$$\forall \vec{w}_0 \gg 0 \quad c^*(\vec{w}_0, y) \text{ exists, and is equal to } \lim_{\vec{w}\to\vec{w}_0} c^*(\vec{w}, y)$$

Theorem 1.23. *(Shephard's Lemma) If $c^*(\vec{w}, y)$ is differentiable at $\vec{w}, y$ and $w_i > 0$ for $i = 1, \ldots, n$ then*
$$x_i^*(\vec{w}, y) = \frac{\partial c^*(\vec{w}, y)}{\partial w_i}$$

1.3.11 Weak Axiom of Cost Minimisation (WACM)

Gives a necessary condition on data to stem from a cost minimising (and thus rationally operating firm)

Given observations of prices $\vec{w}^t \in \mathbb{R}^n_{\geq 0}$, inputs $\vec{x}^t \in \mathbb{R}^n_{\geq 0}$ and outputs $y^t \geq 0$ at time points $t = 1, \ldots, T$. The WACM states that
$$\vec{w}^t(\vec{x}^t)^T \leq \vec{w}^t(\vec{x}^s)^T \quad \forall s,t = 1,\ldots,T \text{ such that } y^t \leq y^s$$
As a corollary
$$(\vec{w}^t - \vec{w}^s)(\vec{x}^t - \vec{x}^s)^T \leq 0 \quad \forall s,t = 1,\ldots,T \text{ such that } y^t \leq y^s$$

1.3.12 Long-run and short-run costs

Let $F, V \subseteq \{1,\ldots,n\}$ be index sets such that $F \cap V = \emptyset$ and $F \cup V = \{1,\ldots,n\}$.

V te indices of variable short-run factors and F comprises the indices of fixed long-run factors.

Take $\underline{x} \in \mathbb{R}^n_{\geq 0}$ and write $\underline{x} = (\underline{x}_F, \underline{x}_V) \in \mathbb{R}^F_{\geq 0} \times \mathbb{R}^V_{\geq 0}$

Definition 1.24. *Level of the fixed factor will influence both the minimised cost, given by the* ***Short-run cost function***
$$c_s^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^F_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$$
$$\begin{aligned} c_s^*(\underline{w}, \underline{x}_F, y) &= \min_{\underline{x}\in\mathbb{R}^V_{\geq 0} \ s.t\ f(\underline{x}_F, \underline{x}_V)=y} \underline{w}\underline{x}^T \\ &= \underline{w}_F\underline{x}_F^T + \min_{\underline{x}_V\in\mathbb{R}^V_{\geq 0} \ s.t\ f(\underline{x}_F, \underline{x}_V)=y} \underline{w}_V\underline{x}_V^T \end{aligned}$$
and the cost-minimising choices of the variable factors, given by the ***short-run conditional factor demand functions***
$$x_s^* : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^F_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$$
$$x_s^*(\underline{w}, \underline{x}_F, y) = \arg\min_{\underline{x}_V\in\mathbb{R}^V_{\geq 0} \ s.t\ f(\underline{x}_F, \underline{x}_V)=y} \underline{w}\underline{x}^T = \arg\min_{\underline{x}_V\in\mathbb{R}^V_{\geq 0} \ s.t\ f(\underline{x}_F, \underline{x}_V)=y} \underline{w}_V\underline{x}_V^T$$
Note that
$$c^*(\underline{w}, y) \leq c_s^*(\underline{w}, \underline{x}_F, y)$$

1.3.13 Average and marginal costs

Assuming the costs $\underline{w} = (\underline{w}_F, \underline{w}_V)$ to be fixed.

Definition 1.25. ***Short-run average cost*** *$SAC(y)$ to be the per-unit cost of producing y units of output:*
$$SAC(y) = \frac{c_s^*(\underline{w}, \underline{x}_F, y)}{y}$$

Assume firm is cost-minimising $\underline{x}_V = \underline{x}_s^*(\underline{w}, \underline{x}_F, y)$

$$SAC(y) = \underbrace{\frac{\underline{w}_V \underline{x}_F^T}{y}}_{SAFC(y)} = \underbrace{\frac{\underline{w}_V \underline{x}_s^*(\underline{w}, \underline{x}_F, y)^T}{y}}_{SACV(y)}$$

SAFC = ***short-run average fixed costs*** *and SAVC* = ***short-run average variable costs***

Definition 1.26. ***Short-run marginal cost*** *$SMC(y)$ to be the change in the per-unit cost of producing y units of output:*

$$SMC(y) = \frac{\partial c_s^*(\underline{w}, \underline{x}_F, y)}{\partial y}$$

In the long-run only have variable input factors i.e $V = \{1, \ldots, n\}, F = \varnothing$ leading to $\underline{x} = \vec{x}_V$ so define the long-run average and marginal costs accordingly

$$LAC(y) = \frac{c^*(\underline{w}, y)}{y}$$

$$LMC(y) = \frac{\partial c^*(\underline{w}, y)}{\partial y}$$

1.3.14 Geometry of costs

FIGURES HERE

Minimum of SAC

$$\frac{\partial}{\partial y} SAC(y) = 0 \iff \frac{\partial c_s^*(\underline{w}, \underline{x}_F, y)}{\partial y} = \frac{c_s^*(\underline{w}, \underline{x}_F, y)}{y}$$

At its local minimum the marginal cost is equal to the average cost.

- $SMC(y) < SAC(y) \iff$ short-run average costs are decreasing in y
- $SMC(y) = SAC(y) \iff$ short-run average costs are increasing in y

As $y \to 0$, short-run average costs explodes in presence of fixed costs, but short-run average variable costs and short-run marginal costs are equal.

Note:

- Area under marginal cost curve (MC) gives total variable costs
- $SMC(y) = SAVC(y)$ at local minimum of $SAVC(y)$
- $SMC(y) \gtrless SAVC(y) \iff SAVC(y)$ is increasing/decreasing in y

FINISH OFF THIS SECTION PROPERLY

1.3.15 Profit Maximisation given minimised costs

Profit maximisation problem:

$$\arg\max_{y>0}\{py - c_s^*(\underline{w}, \underline{x}_F, y)\}$$

$$\underset{\max}{y} > 0\{py - c_s^*(\underline{w}, \underline{x}_F, y)\}$$

First and second-order conditions for optimal level of output given minimised costs are given by

$$\frac{\partial}{\partial y}\{py - c_s^*(\underline{w}, \underline{x}_F, y)\} = 0 \implies p = SMC(y) \quad \text{(FOCs)}$$

$$\frac{\partial^2}{\partial y^2}\{py - c_s^*(\underline{w}, \underline{x}_F, y)\} \leq 0 \implies \frac{\partial^2 c_s^*(\underline{w}, \underline{x}_F, y)}{\partial y^2} \geq 0 \quad \text{(SOCs)}$$

Definition 1.27 (Shutdown condition)**.** ***Shutdown condition*** *is given by*

$$p < SAVC(y) = \frac{\underline{w}_V \underline{x}_V^*(\underline{w}_V, y)^T}{y}$$

If $p < SAC(y)$ then the firm should shut down as it is making a loss.

Long-run profit-maximising supply for a cost-minimising firm given by y s.t

- $LMC(y) = p$
- *$LMC(y)$ must be increasing in y*
- *$LAC(y) \leq p$ - converse of the shutdown condition in the long-run*

If sno such $y > 0$ exists for given p then firm should choose to go out of business

1.3.16 Profit maximisation for a noncompetitive firm

Now we seek

$$\arg\max_{y \geq 0} \{py - c_s^*(\underline{w}, y)\}$$

Just dropping the $\underline{x}_F$ term
First and second-order conditions for finding a profit-maximising position for a monopolist facing an inverse demand function given by

$$\frac{\partial}{\partial y}\{p(y)y - c_s^*(\underline{w}, y)\} = 0 \implies \frac{\partial p(y)}{\partial y} + p(y) = SMC(y) \quad \text{(FOC)}$$
$$\frac{\partial^2}{\partial y^2}\{p(y)y - c_s^*(\underline{w}, y)\} \leq 0 \implies \frac{\partial^2 c_s^*(\underline{w}, y)}{\partial y^2} \geq \frac{\partial^2 p(y)}{(\partial y)^2} y + 2\frac{\partial p(y)}{\partial y} \quad \text{(SOCs)}$$

Rearrange the first-order condition as

$$p(y) \underbrace{\left[1 + \frac{\partial p(y)}{\partial y}\frac{y}{p(y)}\right]}_{1+\frac{1}{\epsilon_D(y)}}$$

where $\epsilon_D(y) := \epsilon_D(p(y))$ is the price elasticity of demand facing the monopolist

$$p'(y) = \frac{1}{D'(p(y))}$$

Must have that $|\epsilon_D| \geq 1$
Can have profit-maximising $y = 0$; when the losses from setting output according to the above conditions is greater than the fixed costs i.e. when

$$SAVC(y) > p(y)$$

2 Theory of the consumer

2.1 Preferences & Utility

Definition 2.1. ***Consumption bundle*** *- vector of goods and services an individual is willing to consume*

$$\underline{x} = \big(x_1, \ldots, x_n\big) \in \mathbb{R}^n_{\geq 0}$$

Set of possible consumption bundles referred to as the ***consumption set*** *- usually taken to be some closed, convex set*

$$X \subseteq \mathbb{R}^n_{\geq 0}$$

Consumers have preferences between bundles $\underline{x}, \underline{x}' \in X$

- $\underline{x} \preccurlyeq \underline{x}'$ *means that the consumer has a preference for* $\underline{x}'$ *over* $\underline{x}$
- $\underline{x} \prec \underline{x}'$ *means that the consumer strictly prefers* $\underline{x}'$ *over* $\underline{x}$
- $\underline{x} \sim \underline{x}'$ *means that the consumer is indifferent between* $\underline{x}$ *and* $\underline{x}'$

$$\underline{x} \sim \underline{x}' \iff \underline{x} \preccurlyeq \underline{x}' \text{ and } \underline{x}' \preccurlyeq \underline{x}$$

Preference relation satisfies the following properties

- ***Completeness***

$$\forall \underline{x}, \underline{x}' \in X, \underline{x} \preccurlyeq \underline{x}' \text{ or } \underline{x}' \preccurlyeq \underline{x}$$

- ***Reflexivity*** *(follows from completeness)*

$$\forall \underline{x} \in X, \underline{x} \preccurlyeq \underline{x}$$

- ***Transitivity***

$$\forall \underline{x}, \underline{x}', \underline{x}'' \in X, \underline{x} \preccurlyeq \underline{x}' \text{ and } \underline{x}' \preccurlyeq \underline{x}'' \implies \underline{x} \preccurlyeq \underline{x}''$$

Following assumptions useful but not necessary

- ***Continuity***

$$\forall \underline{x} \in X, \text{ the sets } \{\underline{x}' : \underline{x}' \preccurlyeq x\} \text{ and } \{\underline{x}' : \underline{x}' \succcurlyeq x\} \text{ are closed}$$

- ***Weak/ Strong Monoticity***

$$\underline{x} \leq \underline{x}' \implies \underline{x} \preccurlyeq \underline{x}' (\text{weak})$$
$$\underline{x} \leq \underline{x}' \text{ and } \underline{x} \neq \underline{x}' \implies \underline{x} \prec \underline{x}' (\text{strong})$$

- ***Local nonsatiation***

$$\forall \underline{x} \in X, \forall \varepsilon > 0, \exists \underline{x}' \in X \text{ s.t. } \|\underline{x}' - \underline{x}\| < \varepsilon \text{ and } \underline{x}' \succ \underline{x}$$

- ***(Strict) Convexity***

$$\text{Given } \underline{x}, \underline{x}', \underline{x}'' \in X \text{ with } x \preccurlyeq \underline{x}' \text{ and } \underline{x} \preccurlyeq \underline{x}'' \text{ then } \forall \lambda \in [0,1], \underline{x} \preccurlyeq (\prec)\ \lambda \underline{x}' + (1-\lambda)\underline{x}'' \quad (\text{strict})$$

Can easily verify strong monotonicity implies local non-satiation

Definition 2.2. ***Utility function*** *- real-valued function* $u : X \to \mathbb{R}$ *such that*

$$u(\underline{x}') \leq u(\underline{x}) \iff \underline{x}' \preccurlyeq \underline{x}$$

Proposition 2.3 (Properties of utility function)**.** *If underlying preferences are complete, transitive, continuous and (strictly) monotone, the corresponding utility function will be continuous and (strictly) monotone. If preferences are (strictly) convex, the utility function is (strictly) quasi-concave.*

Note function $f : X \to \mathbb{R}$*, where* X *is a convex set, is strictly quasi-concave if for all* $x, y \in X, x \neq y$ *and for all* $t \in (0,1)$

$$f(tx + (1-t)y) > \min\{f(x), f(y)\}$$

2.2 Substitution in demand

Definition 2.4. ***Marginal rate of substitution*** *- for goods* i *and* j

$$MRS_{i,j}(\underline{x}) = \frac{\partial u(\underline{x})/\partial x_i}{\partial u(\underline{x})/\partial x_j}$$

Also define ***marginal utility*** *of good* i *as*

$$MU_i(\underline{x}) = \frac{\partial u(\underline{x})}{\partial x_i}$$

2.3 Budget Restraints, Utility Maximisation and Demand

Definition 2.5. ***Budget Set***

$$B = B_{\underline{p},m} = \{\underline{x} \in X : \underline{p} \cdot \underline{x}^T \leq m\} \subseteq X$$

for fixed budget m with $\underline{p}$ *the vector of prices.*

Want to find $\arg\max_{\underline{x}\in B} u(\underline{x})$

Guaranteed solution exists given u continuous and B closed and bounded, and if $\underline{p} > 0$

Theorem 2.6 (Walras' Law)**.**

$$\max_{\underline{x}\in X} u(\underline{x}) \quad \textit{such that} \quad \underline{p}\underline{x}^T = m$$

Call the budget line

$$\{\underline{x} \in X : \underline{p}\underline{x}^T = m\} \subseteq X$$

Definition 2.7. ***Marshallian demand***

$$\underline{x}^* : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+ \ s.t.\ \underline{x}^*(\underline{p}, m) = \arg\max_{\underline{x}\in\partial B} u(\underline{x})$$

Definition 2.8. ***Indirect utility function***

$$v : \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \ s.t.\ v(\underline{p}, m) = \max_{\underline{x}\in\partial B} u(\underline{x}) = u(\underline{x}^*(\underline{p}, m))$$

Definition 2.9. ***Expenditure function***

$$e(\underline{p}, \cdot) : U_p \to [0, \infty), \quad u \mapsto (\underline{p}, u) \quad s.t. \quad u = v((\underline{p}, e(\underline{p}, u)))$$

Provides minimum level of income required to obtain given level of utility at prices $\underline{p}$
Solution to the optimisation problem

$$\textit{Find} \quad \min_{\underline{x}} \underline{p}\underline{x}^T \ \textit{subject to constraints;} \quad u(x) \geq u$$

Definition 2.10. ***Hicksian demand*** *(Compensated demand)*

$$\underline{x}^*_H : \mathbb{R}^n_{n\geq 0} \times \mathbb{R}^n_{\geq 0}, \quad (\underline{p}, u) \mapsto \underline{x}^*_H(\underline{p}, u) = \arg\min_{\underline{x}\in u^{-1}([u,\infty)} \underline{p}\underline{x}^T$$

Expression for expenditure-minimising consumption bundle in terms of prices and desired utility level.

$$x^*_{H,i}(\underline{p}, u) = \frac{\partial e(\underline{p}, u)}{\partial p_i}$$

2.3.1 Slutsky's Equation

Theorem 2.11. *Under usual regularity conditions*

$$\underbrace{\frac{\partial x^*_j(\underline{p}, m)}{\partial p_i}}_{\textit{Total effect}} = \underbrace{\frac{\partial x^*_{H,j}(\underline{p}, v(\underline{p}, m))}{\partial p_i}}_{\textit{Substitution effect}} - \underbrace{x^*_i(\underline{p}, m)\frac{\partial x^*_j(\underline{p}, m)}{\partial m}}_{\textit{Income effect}}$$

For all $p \gg 0, , m > 0$ *and* $\forall i, j \in \{1, \ldots, n\}$

Lemma 2.12. *Reaction to changes in consumer income*

- *For **normal goods** - increased income leads to increased demand*

$$\frac{\partial x_i^*(\underline{p}, m)}{\partial m} > 0$$

- *For **inferior goods** - increased income leads to decreased demand*

$$\frac{\partial x_i^*(\underline{p}, m)}{\partial m} < 0$$

- *For **Luxury goods** - demand increases more than proportionally to income; income elasticity* > 1
- *For **Necessity goods** - demand increases less than proportionally to income; income elasticity* < 1
- *Say consumer has **homothetic preferences** for a set of goods if demand for each good is proportional to income; income elasticity = 1*
- *For **Ordinary goods**, a decrease in prce will lead to an increase in demand*

$$\frac{\partial x_i^*(\underline{p}, m)}{\partial p_i} < 0$$

- *For **Giffen goods**, a decrease in price will lead to a decrease in demand*

$$\frac{\partial x_i^*(\underline{p}, m)}{\partial p_i} > 0$$

  A Giffen good is always an inferior good.

Part II
Markets and Competition

3 Demand and Supply

Definition 3.1. ***Market demand*** *and* ***Market Supply***
Given a market for a single good with I consumers and J firms

$$\begin{aligned} \textit{Consumer } i \textit{ demand:} \quad & x_i^*(\underline{p}, m_i) \quad \textit{for } i = 1, \ldots, I \\ \textit{Supply curve for firm } j\textit{:} \quad & y_j^*(\underline{p}) \quad \textit{for } j = 1, \ldots, J \end{aligned}$$

Then the market demand and supply are given by

$$X^*(\underline{p}, m_1, \ldots, m_I) = \sum_{i=1}^{I} x_i^*(\underline{p}, m_i)$$

$$Y^*(\underline{p}) = \sum_{j=1}^{J} y_j^*(\underline{p})$$

4 Social Welfare

4.1 Consumers' and Producers' Surplus

Consider fixed income levels; $m_1, \ldots, m_I$
Price change from $p^{(1)} > 0 \to p^{(2)>0}$, assuming WLOG $p^{(1)<p^{(2)}}$

Measure the effect of the price change

$$\sum_{j=1}^{J} \pi_j^*(p^{(2)}) - \pi_j^*(p^{(1)}) = \int_{p^{(1)}}^{p^{(2)}} Y(p)dp$$

Introduce the **producer's surplus at price** $\hat{p}$ as one part of the measure for social welfare measure

$$PS(\hat{p}) = \int_0^{\hat{p}} Y(p)dp$$

Define the **compensating variation**

$$CV(p^{(1)}, p^{(2)}, m_i) = \int_{p^{(1)}}^{p^{(2)}} x_{H,i}^*(p, v_i(p^{(1)}, m_i))dp$$

Define the **consumer surplus** at price $\hat{p}$ as

$$CS(\hat{p}) = \int_{\hat{p}}^{\infty} \sum_{i=1}^{I} x_i^*(p, m_i)dp = \int_{\hat{p}}^{\infty} X(p, m_1, \ldots, m_I)dp$$

Take the sum of consumers' surplus and producers' surplus to get the **commounity surplus** at price $\hat{p}$ - measure of social welfare

4.2 Indirect Taxes and Equilibrium

Indirect tax - one that can be passed on to another party

Tax burden determined by relative price elasticity of supply and demand

- If demand more price-elastic than supply - producer will bear more of the tax burden
- If supply more price-elastic than demand - consumer will bear more of the tax burden having it passed on from the producer

Can impose indirect taxes on production of goods in one of 2 ways

- The tax may be a fixed amount per unit sold - **unit/specific tax**
- May be a percentage of the good's price - **ad valorem tax**

Imposing taxes reduces community surplus.
Difference pre- and post-tax is the **deadweight loss** of the tax.

Community surplus = producers' surplus + consumers' surplus + tax revenue

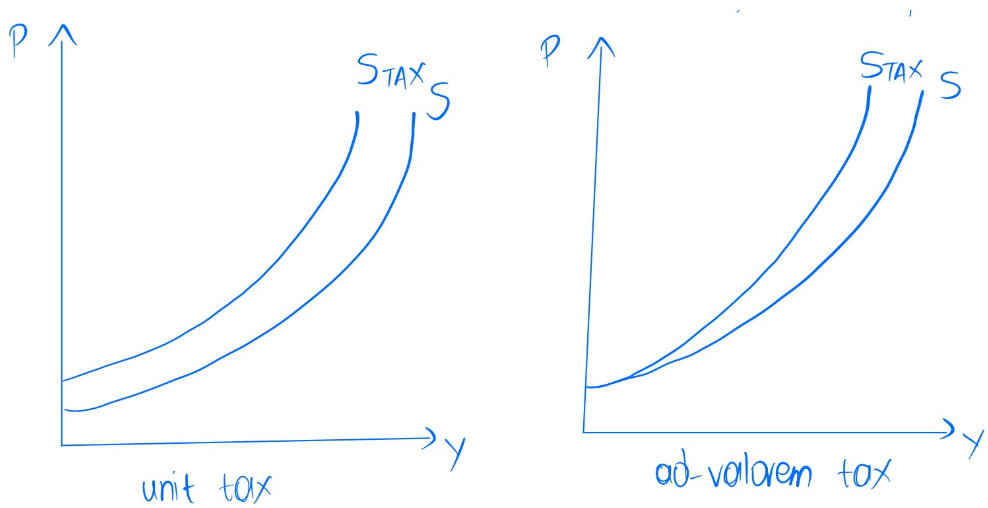
4.2.1 Abnormal profits, Long-run equilibrium and Productive Efficiency

Definition 4.1. ***Accounting costs*** - *include all financial costs of production*

- *All paid costs*
- *Includes fixed and variable costs*

Economic costs - *are accounting costs plus the* ***opportunity costs***

- *Opportunity costs measure the foregone benefit of employing the firm's resources elsewhere, in the best alternative manner*
- *Unit costs for each factor can be defined as economic costs*



- *In long run the opportunity costs should match the accounting profit*

Firms that cover only the economic costs have zero economic profit - but their accounting profit is equal to their opportunity costs; said ot be earning a ***normal profit***
Firms making more than their economic costs are said to be making ***abnormal profits*** *- encourages entry to market by other firms*
Firms making normal profits are said to be ***productively efficient*** *- they produce at the minimum of the average costs curve - taking into account opportunity costs.*
Firms in perfect competition can only make normal profit in the long-run. Due to infinite price-take firms and absence of barriers to entry or exit the market.

Part III
Macroeconomics

4.3 The circular flow of income

Definition 4.2. ***Circular flow of income*** *- the flow of money between households and firms in a closed economy*
2 sectors *- households and firms*

- *Households supply factors of production to firms*
- *Firms use factors of production to produce goods and services*
- *Households consume goods and services*
- *Firms pay wages, rent, interest and profits to households*

Circular flow needs to be in equilibrium

$$\underbrace{C}_{Consumption\ expenditure} = \underbrace{Y}_{Income}$$

5 sector model

- *Households*
- *Firms*

- *Financial sector*
- *Government*
- *Overseas sector*

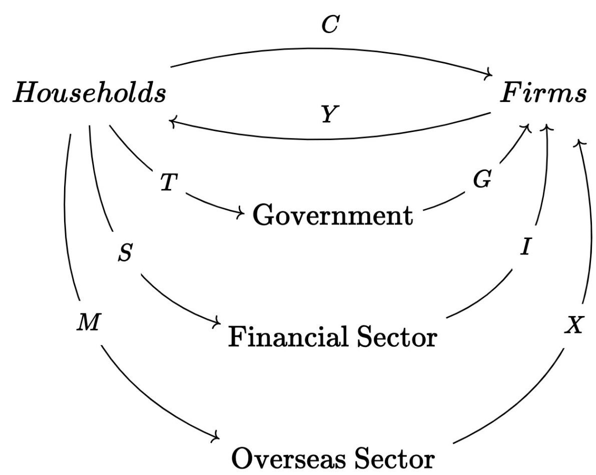


Figure 1: 5 sector model

1. ***Leakages*** *- Not all of a household's income spent on domestic goods and services*
   - *S - Savings (Financial sector)*
   - *T - Taxes (Government)*
   - *M - Imports (Overseas sector)*
2. ***Injections*** *- Money entering the circular flow of income*
   - *I - Investment (Firms)*
   - *G - Government spending (Government)*
   - *X - Exports (Overseas sector)*

Leakages must equal injections for the circular flow to be in equilibrium
Leakages reduce the aggregate demand and injections boost it.
Equilibrium when

$$S + T + M = I + G + X$$

Can define ***aggregate demand*** *as follows*

$$\begin{aligned} AD &= C + I + G + (X - M) \\ &= Injections \ + \ Domestic\ consumer\ spend \end{aligned}$$

Pair the inflows and outflows for the equilibrium condition easily by looking at figure

$$I + G + X = S + T + M$$

- *S and I - financial sector*
- *T and G - government*
- *M and X - overseas sector*

5 Gross Domestic Product (GDP)

To note

- GDP a monetary/nomial value not a real value
- Measures only the gross value added in the course of production
- Measure services provided by state sector at cost
- GNP - gross national product - GDP plus gross output of all citizens abroad

5.0.1 Calculating GDP

1. **Production approach** - Calculate the gross value added in the domestic production. This is gross value of output minus intermediate consumption
2. **Expenditure approach** - "Everything produced has to be bought"

$$Y = C + I + G + (X - M)$$

3. **Income approach** - "Someone has to earn the value that has been created"

5.0.2 Critisism of GDP

1. Since GDP is a nominal value - price changes (due to inflation) can cause an increase in GDP
2. Many services ignored - e.g. housework
3. Externalities often ignored - e.g. pollution
4. Depreciation often ignored - e.g. wear and tear of machinery/infrastructure
5. Ignores the benefits of leisure time

5.0.3 Problems with GDP as a measure of welfare

1. Inflation increases GDP but not welfare
2. Tendency to commercialise services - e.g. childcare/eldercare or voluntary work
3. Increase of industry production despite adverse effects to the environment
4. Might be incentives to cause depreciation in order to re-buil infrastructure. Also in business, might be incentives to produce low quality goods
5. People may be pished into (dependent) work despite their Preferences

5.1 Allocation of income - connections to social welfare

How to describe distribution of income?

- Mean
- Median
- Range
- IQR
- Some quantiles

- Variance/Standard deviation

Definition 5.1. ***Gini coefficient***
For population of n *persons with income* $y_1, \dots, y_n$ *defined as*

$$G = \frac{\sum_{i,j=1}^{n} |y_i - y_j|}{2n \sum_{i=1}^{n} y_i}$$

Can show that the Gini coefficient is

- *Positively homogeneous of degree* 0 *in* $y_1, \dots, y_n$
- *Always between 0 and 1*
- *0 if and only if everyone has the same income*
- *1 if and only if one person has all the income*

5.1.1 Normative positions

- **Equal distribution** - Asserts everyone should have the same income
- **Minimax approach** - Suppose you were born into a society with uncertainty in which class you will be born. The most risk averse approach would then be to minimise the maximally adverse outcome (therefore the name). Equivalently, one would try to maximise the minimal utility in the population
  Criticism: One needs to compare different utility functions.
- **Pareto efficiency** - A Pareto improvement is a change that makes at least one person better off, without making any other person worse of. An allocation is Pareto efficient if no Pareto improvement is possible.
  Even though this is a widely agreed criterion for income allocation, it is also a rather week notion of efficiency. E.g. if utilities are strictly increasing in income, also an allocation is Pareto efficient where one person owns the entire income and the rest does not have anything at all.
- **Care about the outcome of the income allocation - but only about the underlying mechanism.** If the allocation mechanism is fair (if it works according to fair rules and laws), then any resulting allocation is fair. This position is probably most akin to a purely capitalistic approach.