Mathematical Finance: Intro to Option Pricing Concise Notes

MATH60012

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Content from prior years assumed to be known.

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Contents

1 What are Derivatives

1.1 Forward Contract

Definition 1 (Forward Contract)**.** *A **forward contract** is an agreement to buy or sell an asset at a certain future time for a certain price, paying a **forward price** at **expiry** of the contract.*
*Described by parameters A - the **asset** being bought or sold, K - the **strike price** and T - the **expiry date**.*
The value of the contract at time t is $P_T - K$, where P_T the market price of the asset at time T.

$$F_T = f(P_T), \quad \text{with } f(x) := x - K, \quad x \in \mathbb{R}$$

1.2 The Binary Option

Definition 2 (Binary Option)**.** *A **binary option** is a contract that pays out a fixed amount if the price of the underlying asset is above a certain level at expiry, and nothing otherwise.*
*The **payoff** of a binary option is given by*

$$f(P_T) = f(x) := 100 \cdot 1_{x \geq K}, \quad 1_{x \geq K} = \begin{cases} 1 & \text{if } P_T > K \\ 0 & \text{if } P_T \leq K \end{cases} \quad P\&L \text{ options } = \begin{cases} 100 \cdot A - AB_0 & \text{if } P_T > K \\ -AB_0 & \text{otherwise} \end{cases}$$

where B_0 the intial value, receiving payoff of \$100 for each contract held.

1.3 Speculation

Definition 3 (Speculation)**.** ***Speculation** is the act of trading in the hope of making a profit, but with the risk of making a loss.*
***Hedging** is the act of trading to reduce risk.*

2 How to price and hedge derivatives in one-period models

2.1 Pricing by replication

Principle 5. *(Law of One Price)*
If there are 2 possible investments which have ,under all possible market outcomes, the same value at time T , then they must have the same value also at all previous times.

Principle 6. *(Pricing via replication)*
At any time $t \in [0, T]$, the derivative's price must equal the value of the replicating portfolio.

Remark 7. *(Hedging = - Replicating)*

2.2 Pricing a Binary Option

Assume in the market we can only

- Trade gold, or
- Borrow/deposit money from/into a bank

Over a short-time span can assume that interest rate $r = 0$.
We trade binary options with strike price $K = 59$ and expiry $T = 1$, and the current price of gold is $P_0 = 58$.

$$\underbrace{P_T(\omega)}_{\text{Price at maturity}} = \begin{cases} 61, & \text{if } \omega = g; \\ 56, & \text{if } \omega = b. \end{cases}, \quad \Omega = \{g, b\}, \quad \mathbb{P}(\{g\}) = \mathbb{P}(\{b\}) = \frac{1}{2}$$

With the payoff $B_T = 100 \cdot 1_{P_T \geq 59}$

$$B_T(\omega) = \begin{cases} 100, & \text{if } \omega = g; \\ 0, & \text{if } \omega = b. \end{cases}$$

Price now by replication

$$\underbrace{V_T}_{\text{wealth at maturity}} = V_T^{x,h} = x - h \cdot 58 + h \cdot P_T$$

where h the amount of gold bought initially at P_0 and $V_0 = x$ the initial wealth.
Replicating if final wealth is the same as the payoff

$$\begin{cases} x + h \cdot (61 - 58) = 100 \\ x + h \cdot (56 - 58) = 0 \end{cases} \implies h = 20, x = 40$$

So we get the initial value of the binary option is $V_0 = 40$.

2.3 Model uncertainty

Definition 8 (Model uncertainty)**.** ***Model uncertainty*** *is the uncertainty about the probability distribution of the future asset price.*
Introduce model risk - the risk that the model we use to price the derivative is not the correct one.

Remark 9 (Storage Costs)**.** *In the real world, there are storage costs for holding the asset for some assets. e.g. gold, oil, etc. but not for others e.g. stocks, bonds, etc.*

Remark 10 (Choosing the right mode)**.** *In the real world, we do not know the true model.*
We choose the model that best fits the data, and use it to price the derivative.

Remark 11 (Counter-party risk)**.** *In the real world, there is counter-party risk - the risk that the other party in the contract will not be able to pay.*

2.4 What can be used as underlying

- Commodities
- Bonds
- Stocks
- Stock indices
- FX rates
- Other derivatives

2.5 The bank account and the bond

Assume that we can invest in contract - with value B_t at time t given by

1. $B_0 = 1$, by normalisation
2. If discrete time: $B_{t+1} = B_t(1 + R_t)$, where R_t the interest rate for period $(t, t+1)$

$$B_t = B_0 \prod_{s=0}^{t-1} (1 + R_s)$$

   if $R_t = r$ constant then, $B_t = B_0(1 + r)^t$
3. If continuous time: satisfy $dB_t = B_t R_t dt$, where R_t the instantaneous spot rate at t . Then $B_t = B_0 e^{\int_0^t R_s ds}$, if $R_t = r$ constant then $B_t = B_0 e^{rt}$

2.6 Justification of the law of one price

Remark 12 (Transaction costs and arbitrage). *In the real world, there are transaction costs. These costs nay be higher than the price imbalance that we are trying to exploit - hence no arbitrage.*

2.7 The no-arbitrage principle

Definition 13 (Arbitrage). ***Arbitrage*** *is the act of making a profit without risk.*
Arbitrage opportunity *is a trading strategy that has a positive initial value and zero probability of a loss. It is a portfolio L that starting with* 0 *initial capital, and final value* V_T^L *s.t* $V_T^L \geq 0$ *a.s., and* $\mathbb{P}(V_T^L < 0) = 0, \mathbb{P}(V_T^L > 0) > 0$

- *If there exists arbitrage, law of supply and demand would quickly disappear it.*
- *If the law of one price fails, then there exists arbitrage*

Proposition 14. *(Arbitrage and the Binomial model)*
Given market with one risky asset S *the stock and the bond. We assume that the interest rate is* $r > -1$ *and the stock price can only take two values at time* T, $S_T = S_0 u$ *or* $S_T = S_0 d$ *with* $0 < d < 1 < u$.
Then there is no arbitrage if and only if $d < 1 + r < u$.

2.8 Dependence of prices on probabilities

Definition 15 (Equivalent probabilities). *Say* $\mathbb{P} \sim \mathbb{Q}$ *are equivalent if they have the same null sets.*

2.9 Unspoken Modelling assumptions

Remark 16 (Linear dependence). *Given discrete time, and some model for the bonds and shares prices, with stochastic processes* K, H *represent the number of shares and bonds held between* $(t, t+1)$, *with* H *vector valued quantity* $H = (H^1, \ldots, H^m)$ *with* H^j *the number of shares in the stock of type* j *of price* S^j, *then we take*

$$V_s^{(K,H)} := K_t B_s + H_t \cdot S_s, \quad \text{for } s = t, t+1$$

the value of portfolio (K, H) *where* $S = (S^1, \ldots, S^m)$ *the stock prices.*
Assume that $(K, H) \mapsto V^{(K,H)}$ *is linear.*

1. *Anything behaves linearly only does so on first approximation*
2. *Linear models easy to work with, and often only ones that are solvable.*

2.10 Prices of liquid and illiquid goods

Important to distinguish between illiquid markets and liquid ones; the latter those in which a good is traded a lot, by many possible participants - cannot use law of one price in illiquid markets.
Options pricing deals with how to price illiquid derivatives based on a liquid underlying.

2.11 Modelling the underlying, not the derivative

2.12 Discounting and Numeraire

> *"A dollar today is worth more than a dollar tomorrow."*

Definition 17. *(Nominal vs. Real value)*
Look at value of W_t *in terms of* B *the bond instead of a unit of currency.*

$$\underbrace{\overline{W}_t}_{\text{real terms}} = \underbrace{W_t}_{\text{nominal terms}} / B_t$$

Given constants $B_0 = 1, B_1 = B_0(1+r) > 0, r > -1, S_0 \in \mathbb{R}^m$ and the random vector $S_1 \in \mathbb{R}^m$
Consider portfolio (k,h) with value $V_s^{k,h} = kB_s + h \cdot S_s$ at time $s \in \{0,1\}$, more interested in initial capital - look at (x,h) instead with $x = k + S_0 h$ the initial capital and h^j number of shares of stock S^j
Value of (x,h) is

$$V_0^{x,h} = x, \quad V_1^{x,h} = x(1+r) + h \cdot (S_1 - S_0(1+r)), \quad (x,h) \in \mathbb{R} \times \mathbb{R}^m$$

Since $\overline{V}_t^{x,h} = V_t^{x,h}/B_t$ find that $\overline{V}_0^{x,h} = V_0^{x,h} = x$
So our value in discounted terms becomes

$$\overline{V}_1^{x,h} = \frac{1}{1+r}(x(1+r) + h \cdot (S_1 - S_0(1+r))) = x + h \cdot \left(\frac{S_1}{1+r} - S_0\right)$$

$$\overline{V}_t^{x,h} = x + h \cdot (\overline{S}_1 - \overline{S}_0), \quad t = 0, 1$$

In discounted terms the gains from trade between times $0, t$ given by

$$\overline{V}_t^{x,h} - \overline{V}_0^{x,h} = h \cdot (\overline{S}_1 - \overline{S}_0)$$

$$\textit{Continuous case: } d\overline{V}_t^{x,h} = H_t \cdot d\overline{S}_t, \qquad \overline{V}_t^{x,h} - \overline{V}_0^{x,h} = \int_0^S H_t \cdot d\overline{S}_t$$

Definition 20 (Numeraire)**.** *A numeraire is a portfolio L which has a strictly positive value a.s. and at all times, ie.e s.t. $V_t^L > 0$ a.s. for all t*

2.13 Finite Probability Spaces

Pretty straightforward stuff here

2.14 How to find arbitrage

Trading strategy $(x,h) \in \mathbb{R} \times \mathbb{R}^m$ is an arbitrage if $x = 0$ and

$$\mathbb{P}(\overline{V}_1^{0,h} \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(\overline{V}_1^{0,h} = 0) < 1$$

or equivalently

$$\mathbb{P}(\overline{V}_1^{0,h} < 0) = 0 \quad \text{and} \quad \mathbb{P}(\overline{V}_1^{0,h} > 0) > 0$$

Since x taken to be zero, say 'h is an arbitrage'.
h is an arbitrage if $w^h \in \mathbb{R}^n$ corresponding to random variable $\overline{V}_1^{0,h} = h \cdot (\overline{S}_1 - \overline{S}_0)$, satisfies $w_i^h \geq 0, \ \forall i$ and $w^h \neq 0$
Denote W the vector space of discounted payoffs replicable at cost 0

$$W := \left\{ \sum_{j=1}^m h^j (\overline{S}_1^j - \overline{S}_0^j) : h \in \mathbb{R}^m \right\} = \operatorname{span}\left\{ (\overline{S}_1^j - \overline{S}_0^j)_{j=1}^m \right\}$$

Find that

The set of all arbitrage payoffs is $W \cap (\mathbb{R}_+^n \setminus \{0\})$

No arbitrage iff $W \cap (\mathbb{R}_+^n \setminus \{0\}) = \{0\}$
The set of discounted payoffs replicable at cost x is $x + W = \{x + w : w \in W\}$, since $\overline{V}_1^{x,h} = x + \overline{V}_1^{0,h}$

2.15 The Fourier-Motzkin Algorithm

2.15.1 Simple linear equality

Given $x \in \mathbb{R}^n, b \in \mathbb{R}^m$ and a $m \times n$ matrix A with rows, $a^1, \dots, ^m$, to solve system $Ax = b$ write $x = (y, z)$ with $y = (x_1, \dots, x_{n-1} \in \mathbb{R}^{n-1}, z := x_n \in R)$ so that $a^i \cdot x = c^i \cdot y + d^i z$ for some $c^i \in \mathbb{R}^{n-1}, d^i \in \mathbb{R}$ rewriting the equations as

$$c^i \cdot y + d^i z = b_i, i = 1, \dots, m$$

in the form $z = e^i \cdot y + f^i$ or in form $0 = e^i \cdot y + f^i$ for some $e^i \in \mathbb{R}^{n-1}, f^i \in \mathbb{R}$ to find the equivalent system

$$\begin{cases} z = e^i \cdot y + f^i, & \text{for } i \in I_{\neq} := \{i : d^i \neq 0\}; \\ 0 = e^i \cdot y + f^i, & \text{for } i \in I_{=} := \{i : d^i = 0\}. \end{cases}$$

$I_{\neq}, I_{=}$ disjoint subsets whose union is $\{1, \dots, m\}$.
$I_{\neq} = \varnothing$ iff the last system i.e $Ax = b$ doesn't involve variable z
In this case: take any y solving system $0 = e^i \cdot y + f^i, i \in I_{=}$ and any $z \in \mathbb{R}$
If instead $I_{\neq} \neq \varnothing$ then fix arbitrarily $i^* \in I_{\neq}$ then all and only the solutions (y, z) of above found by taking solution y of system

$$\begin{cases} e^i \cdot y + f^i = e^j \cdot y + f^j, & \text{for } i, j \in I_{\neq}; \\ 0 = e^i \cdot y + f^i, & \text{for } i \in I_{=}. \end{cases}$$

and taking $z = e^{i^*} \cdot y + f^{i^*}$
Above equation has no solution iff so does $Ax = b$

Iterating the above get sequence of linear systems $A^i x^i = b^i, i = n, n-1, \dots, 1$
Deleting the last variable, one at a time, till we get to $A^1 x^1 = b^1$ in the only variable $x^1 = x_1 \in \mathbb{R}$
$A^1 x^1 = b^1$ has no solution $\iff$ so does $Ax = b$
Can construct all solution of $Ax = b$ from solutions of $A^1 x^1 = b^1$ by back substitution.

2.15.2 System of linear inequalities

Generalise above to system of linear inequalities $Ax \geq b$
Rewriting our system instead as

$$\begin{cases} z \geq e^i \cdot y + f^i, & \text{for } i \in I_{>} : \{i : d^i > 0\}; \\ z \leq e^i \cdot y + f^i, & \text{for } i \in I_{<} : \{i : d^i < 0\}; \\ 0 \geq e^i \cdot y + f^i, & \text{for } i \in I_{=} : \{i : d^i = 0\}; \end{cases}$$

where $I_{<}, I_{>}, I_{=}$ disjoint sets whose union is $\{1, \dots, m\}$, with $I_{<} \cup I_{>} = \varnothing$ iff the last system does not involve variable z
In this case: take any y solving system $0 \geq e^i \cdot y + f^i, i \in I_{=}$ and any $z \in \mathbb{R}$
If instead $I_{<} \cup I_{>} \neq \varnothing$, if $I_{>} = \varnothing$ then $I_{<} \neq \varnothing$ and all and only solutions (y, z) of above found by taking solution y of system $0 \geq d_i + f_i \cdot y, i \in I_{=}$ if $I_{=} \neq \varnothing$ and taking any $z \leq \min_{i \in I_{<}} d_i + f_i \cdot y$
Analogously if $I_{<} = \varnothing$ then $I_{>} \neq \varnothing$ and take any y solving $0 \geq d_i + f_i \cdot y, i \in I_{=}$ if $I_{=} \neq \varnothing$, and any $z \geq \max_{i \in I_{>}} d_i + f_i \cdot y$ and find all and only solutions of the system .

In case where $I_{>}, I_{<}$ both non-empty, all and only solutions $x = (y, z)$ obtained by: taking solution y of system

$$\begin{cases} e^i \cdot y + f^i \geq e^j \cdot y + f^j, & \text{for } i \in I_{<}, j \in I_{>}; \\ 0 \geq e^i \cdot y + f^i, & \text{for } i \in I_{=}, \end{cases}$$

and then choose any

$$z \in \left[\max_{j \in I_{>}} e^j \cdot y + f^j, \min_{i \in I_{<}} e^i \cdot y + f^i \right].$$

Note conventions, $\sup \varnothing = -\infty, \inf \varnothing = \infty$
When $I_> = \varnothing$ or $I_< = \varnothing$ then system becomes $z \leq \min_{i \in I_<} d_i + f_i \cdot y$, $z \geq \max_{i \in I_>} d_i + f_i \cdot y$ respectively.
And if $I_< \cup I_> = \varnothing$ then we have $z \in \mathbb{R}$
(y, z) solves system iff y solves the rewritten system and z solves the above range.

Iterating this procedure get sequence of linear systems $A^i x^i \geq b^i, i = n, n-1, \ldots, 1$
Where i^{th} system has variables $x^i = \left(x_1, \ldots, x_i\right)$, starting from x^n and deleting until we get to $x^1 = x_1 \in \mathbb{R}$
$A^1 x^1 \geq b^1$ has no solution $\iff$ so does $Ax \geq b$
Can construct all solution of $Ax \geq b$ from solutions of $A^1 x^1 \geq b^1$ by back substitution.

2.16 Find arbitrage with the FM algorithm

Can find if a finite market has arbitrage, can use FM algorithm to check whether $W \cap \mathbb{R}^n_+ = \{0\}$

2.17 The no-arbitrage and the domination principles

Principle 32 (Weak Domination Principle)**.** *If there are 2 possible investments, and the first one has, under all possible market outcomes, a smaller value then hte second at time T, then this holds also at all previous times.*

Principle 33 (Pricing via super- and sub-replication)**.** *At any time $t \in [0, T]$, the derivative's price must be smaller (resp. bigger) than the value of any super-replicating (resp. sub-replicating) portfolio.*
We can then define the price bounds for the derivative as the smallest (resp. biggest) initial capital of a super-replicating (resp. sub-replicating) portfolio.

$$u(X) := \inf \left\{V_0^U : V_T^U \geq X_T\right\}, \quad d(x) = \sup \left\{V_0^D : V_T^D \leq X_T\right\}$$

We also have trivially $V_T \geq X_T \iff \overline{V}_T \geq \overline{X}_T$ and $V_T \leq X_T \iff \underline{V}_T \leq \underline{X}_T$

$$u(X) = \inf \left\{V_0^U : \overline{V}_T^U \geq \overline{X}_T\right\}, \quad d(X) = \sup \left\{V_0^D : \overline{V}_T^D \leq \overline{X}_T\right\}$$

Principle 34 (Domination Principle)**.** *Say that the Strict Domination Principle holds if, whenever there are two investments L, M such that*

1. *the first one has a value a.s. smaller than the second, i.e. $\mathbb{P}(\{V_t^L \leq V_t^M\}) = 1$*
2. *the first one has a value not a.s. equal to the second i.e. $\mathbb{P}(\{V_t^L \neq V_t^M\}) > 0$*

at time $t = T$, then necessarily items 1 and 2 hold at all previous times $t \in [0, T]$.
We will say that the Domination Principle holds if both the law of one price and the strict domination principle hold.

Remark 35. *When working in one-period models, items 1 and 2 in principle in principle 34 are just required to hold for $t = 0$. Since V_0 is known at time 0, it is a constant, so items 1 and 2 become more simply $V_0^L < V_0^M$*

Theorem 36. *In the linear one-period marker model, the following are equivalent:*

1. *The domination principle (34) holds*
2. *The strict domination principle holds*
3. *there exists no-arbitrage*

Theorem 38. *In the linear multi-period market model, the Weak Domination principle (32) holds if and only if there exist no uniform arbitrage.*
Moreover, the domination principle (34) implies the weak domination principle (32), which implies teh Law of one price, adn the two opposite implications do not hold.

2.18 No-arbitrage prices

Definition 39 (No-arbitrage price)**.** *$p \in \mathbb{R}$ is a fair price of X in the market (B, S) if the enlarged market (B, S, X), composed of the original market plus the derivative X traded has price $X_0 = p$ at time 0, is also arbitrage-free.*

Lemma 40. *The infimum and supremum in the definition of $u(X)$ and $d(X)$ are achieved.*

Proposition 42. *In the arbitrage-free one-period market (B, S), if a derivative X is*

1. *replicable, then its fair price X_0 is unique, it equals the initial value x of any replicating portfolio, and $u(X) = X_0 = d(X)$*
2. *not replicable, then the set of its fair prices is the open interval $(d(X), u(X))$*

Definition 44. *A market model (B, S) is called complete if any derivative X is replicable (in such a market); otherwise its called incomplete.*
In a complete model all derivatives have a unique price.

2.19 Linear programming, option pricing and arbitrage

Definition 45. *A set $P \subseteq \mathbb{R}^k$ is called a polyhedron if it of the form*

$$P := \{z \in \mathbb{R}^k : Az \geq b, \quad Cz \leq d, \quad Ez = f\}$$

where A, C, E are matrices and b, d, f are vectors. A linear optimisation problem (LP) is a problem of the form

$$\textit{minimise/maximise} a + c \cdot z \textit{ subject to } z \in P$$

where $a \in R, c, z$ are vectors and P is a polyhedron.
Polyhedra are closed, convex sets.

2.20 Solving LPs with the FM algorithm

Given a polyhedron $P \subseteq \mathbb{R}^n$ and $c \in \mathbb{R}^n$, and the LP

$$\inf c \cdot z \text{ subject to } z \in P$$

Want to find z^* such that $c \cdot z^* = y^*$ the infimum
z^* the optimiser, y^* the optimal value.
If optimiser exists, LP called *solvable*

To solve the LP, define the polyhedron

$$R = \{(y, z) : y = c \cdot z, z \in P\}$$

Use FM to compute projection $\pi_1^{n+1}(R)$ where $\pi_1^{n+1} = \pi_y$ defined on $\mathbb{R}_y^1 \times \mathbb{R}_y^n$

Theorem 47. $\inf \pi_1^{n+1}(R)$ *equals the optimal value y^* of the LP, and z^* solves the LP if and only if $(y^*, z^*) \in R$*

Corollary 48. *A LP solvable iff its optimal value is finite.*

2.21 How to price a derivative using the FM algorithm

2.22 How to find the optimisers of a LP using the FM algorithm

2.23 Foreign Currency

Describe the money market using 'the bond' and 'the stock'
If we are trading between two currencies, need to consider

$$B_0^d = 1, \quad B_0^f = 1, \quad B_0^d = 1 + r^d, \quad B_0^f = 1 + r^f$$

Where B^d the domestic bond, and B^f the foreign bond, each with their own interest rate > -1 .
Also required to specify between domestic and foreign assets each priced in their own currency.
A European stock has price (in €) S , so its price in £ is SE where $E := E_{£}^{€}$ - the cost of one € in £.
Conversely U a British stock its value UE' in €, where $E' := E_{€}^{£}$ - the cost of one £ in €.
Where obviously $E = \frac{1}{E'}$

A british investor may model the market as (B^d, S^d, EB^f, ES^f) where as a euro investor may model the market as $(B^d/E, S^d/E, B^f, S^f)$, with each using their respective bond as the numeraire.
Clear to see that the two models are equivalent, one is arbitrage free/complete iff the other is too.

2.24 Formulas for the binomial model

Consider the one-period binomial model (B, S) assume that $S_0 d = S_1(T) < S_1(H) = S_0 u$
Do not assume that $d < 1 + r < u$, (arbitrage free)
To price a derivative, try to replicate it, i.e find x, h such that

$$\overline{V}_1^{x,h}(H) = \overline{X}_1(H), \quad \overline{V}_1^{x,h}(T) = \overline{X}_1(T)$$

We have $\overline{V}_1^{x,h} = x + h(\overline{S}_1 - \overline{S}_0)$, this system of two equations in two unknowns has a unique solution.

We get the following delta-hedging formula

$$h = \frac{\overline{X}_1(H) - \overline{X}_1(T)}{\overline{S}_1(H) - \overline{S}_1(T)}$$

and the following formula for $\widetilde{p}$ the risk-neutral probability

$$\widetilde{p} = \frac{1 + r - d}{u - d}$$

And the risk-neutral price

$$\overline{X}_0 = \widetilde{p}\overline{X}_1(H) + (1 - \widetilde{p})\overline{X}_1(T), \quad X_0 = \frac{1}{1 + r} E[X_1]$$

2.25 The Fundamental theorem of Asset Pricing

A probability space $(\Omega, \mathcal{A}, \mathbb{P})$, if $\mathbb{Q}$ another probability on $(\Omega, \mathcal{A})$
Say $\mathbb{Q}$ *absolutely continuous* with respect to $\mathbb{P}$, $\mathbb{Q} \ll \mathbb{P}$ if $\mathbb{P}(A) = 0 \implies \mathbb{Q}(A) = 0, \forall A \in \mathcal{A}$
If any null set of $\mathbb{P}$ a null set of $\mathbb{Q}$ say they are equivalent. $\mathbb{Q} \sim \mathbb{P}$ if $\mathbb{Q} \ll \mathbb{P}, \mathbb{Q} \gg \mathbb{P}$

If Ω finite and every singleton $\{w\}, w \in \Omega$ is measurable, then $\mathbb{Q} \ll \mathbb{P}$ iff $\mathbb{P}(\{w\}) = 0 \implies \mathbb{Q}(\{w\}) = 0$
If Ω finite and $\mathbb{P}(\{w\}) > 0, \forall w \in \Omega$ then any probability $\mathbb{Q}$ satisfies $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{Q} \sim \mathbb{P}$ iff $\mathbb{Q}(\{w\}) > 0, \forall w \in \Omega$

Proved earlier that the one-period binomial model arbitrage free iff there exists $\mathbb{Q}$ on $\{H, T\}$ that is equivalent to $\mathbb{P}$ and such that $\overline{S}_0 = \mathbb{E}^{\mathbb{Q}}\overline{S}_1$ holds

Definition 58. *A probability $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is an $\overline{S}$-Martingale Measure, if $\mathbb{Q} \ll \mathbb{P}$ and*

$$\overline{S}_1^j \in L^1(\mathbb{Q}), \quad \overline{S}_0^j = \mathbb{E}^{\mathbb{Q}}[\overline{S}_1^j], \quad \forall j = 1, \ldots, m \tag{53}$$

Denote

$$\mathcal{M}(\overline{S}) := \{\mathbb{Q} \sim \mathbb{P} : \text{ eq. (53) holds}\}, \quad \mathbb{M}(\overline{S}) := \{\mathbb{Q} \ll \mathbb{P} : \text{ eq. (53) holds}\}$$

We have the families of all EMM (Equivalent Martingale Measures) and MM (Martingale measure) for $\overline{S}$ abbreviated as $\mathcal{M}, \mathbb{M}$

Remark 59 (MM in finite Ω)**.** *Any random variable defined on a finite Ω is integrable with respect to any probability; thus, if Ω is finite, the assumption $\overline{S}_1^j \in L^1(\mathbb{Q})$ automatically satisfied. Moreover, because of our assumption that $\mathbb{P}(\{w\}) > 0, \forall w \in \Omega, \mathbb{Q} \ll \mathbb{P}$ is satisfied by any probability $\mathbb{Q}$, and $\mathbb{Q} \sim \mathbb{P}$ holds iff $\mathbb{Q}(\{w\}) > 0 \forall w \in \Omega$*

Theorem 63 (1st Fundamental Theorem of Asset Pricing)**.** *Consider the linear one-period market model (B, S). This model is free of arbitrage $\iff \mathcal{M}(\overline{S}) \neq \varnothing$*

Theorem 64 (Seperation Theorem)**.** *If $C, K \subseteq \mathbb{R}^n$ are convex, C is closed and K is compact then there exists $z \in \mathbb{R}^n, a, b \in \mathbb{R}$ s.t*

$$x \cdot z \leq a < b \leq y \cdot z, \quad \forall x \in C, \forall y \in K$$

In particular, if C is a vector space then $x \cdot z = 0, \forall x \in C$

Theorem 66. *Consider the linear one-period model (B, S) and assume that Ω is finite. In this model there exists no uniform arbitrage $\iff \mathbb{M}(\overline{S}) \neq \varnothing$*

2.26 The Risk Neutral Pricing Formula

Corollary 67 (Risk-Neutral Pricing Formula)**.** *The set of arbitrage-free prices for an illiquid derivative with payoff X_1 in a one-period arbitrage-free market model (B, S) is*

$$\mathcal{AFP}(X_1) = \{\mathbb{E}^{\mathbb{Q}}[\overline{X}_1] : \mathbb{Q} \in \mathcal{M}(\overline{S}) \text{ and } \mathbb{E}^{\mathbb{Q}}[\overline{X}_1] < \infty\}$$

Corollary 69 (Risk-Neutral Pricing Formula)**.** *If (B, S) a one-period market model with no uniform arbitrage, based on a finite probability space, the set of prices for an illiquid derivative with payoff X_1 for which the extended market (B, S, X) has no uniform arbitrage equals*

$$\{\mathbb{E}^{\mathbb{Q}}[\overline{X}_1] : \mathbb{Q} \in \mathbb{M}(\overline{S}) \text{ and } \mathbb{E}^{\mathbb{Q}}[\overline{X}_1] < \infty\}$$

Remark 70. *It follows from Corollary 68 and Proposition 42 that the price bounds for X are given by*

$$u(X) = \sup\{\mathbb{E}^{\mathbb{Q}}(\overline{X}_1) : \mathbb{Q} \in \mathcal{M}(\overline{S}), \mathbb{E}^{\mathbb{Q}}[\overline{X}_1] < \infty\}$$
$$d(x) = \inf\{\mathbb{E}^{\mathbb{Q}}[\overline{X}_1] : \mathbb{Q} \in \mathcal{M}(\overline{S}), \mathbb{E}^{\mathbb{Q}}[\overline{X}_1] < \infty\}$$

sup and inf only attained if and only if X_1 replicable.

Corollary 73 (Characterisation of replicable derivatives)**.** *X_1 is replicable $\iff \mathbb{E}^{\mathbb{Q}}[\overline{X}_1]$ is constant across all $\mathbb{Q} \in \mathcal{M}(\overline{S})$ s.t. $\mathbb{E}^{\mathbb{Q}}[\overline{X}_1] < \infty$*

Corollary 74. *Let (B, S) be a market-model free of arbitrage. Then (B, S) is complete $\iff$ the EMM is unique (i.e. $\mathcal{M}(\overline{S})$ a singleton)*

2.27 Dividends

Stocks can issue dividends, and bonds coupons. Dividends generally paid every 3 months in the same amount. Paying out dividends decreases the value of the shares by the same amount.
Denote with $\underbrace{S}_{\text{ex/post-dividend}}$ the price of share just after the dividend D is paid, and $\underbrace{D}_{cum-dividend}$ the price of one share just before the dividend is paid. Then clearly

$$V = S + D, \quad S \geq 0, D \geq 0$$

Calculate the forward price F of a stock paying dividends in one-period model. F depends on D, forward contract pays $S_1 - F$ but does not depend on D, the stock does pay dividends
If we buy one share at cost S_0, while borrowing $\frac{F+D}{1+r}$ from the bank, we replicate the payment of $S_1 - F$ of the forward contract, since the final wealth is

$$(S_1 + D) - (1 + r)\left(\frac{F + D}{1 + r}\right) = S_1 - F$$

Since the initial cost of such replicating strategy is $S_0 - \frac{F+D}{1+r}$, setting this to 0 shows that the forward price of S is $F = S_0(1 + r) - D$

3 Multi-Period models

3.1 Measurability

Stochastic processes: a family $X = (X_t)_{t \in I}$, i.e. a function $I \ni t \mapsto X_t$ of random numbers or vectors defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ indexed by set I
Say $X \leq Y$ mean that $X_t \leq Y_t$ for all $t \in I$

$$P(\{\omega : X_t(\omega) \leq Y_t(\omega) \forall t \in I\}) = 1 \quad P(\{\omega : X_t(\omega) \leq Y_t(\omega)\}) = 1, \quad \forall t \in I$$

Lemma 78 (Doob-Dynkin)**.** *Suppose X, Y random vectors with n, k components, defined on measurable space $\Omega, \mathcal{A}$, then the following are equivalent*

1. *There exists a Borel function $f : \mathbb{R}^k \to \mathbb{R}^n$ such that $X = f(Y)$*
2. *X is $\sigma(Y)$-measurable, i.e. $\sigma(X) \subseteq \sigma(Y)$*

Lemma 79. *Given random vector X, Y on the same measurable space $\Omega, \mathcal{A}$ if Y only takes countably many values $\{y_n\}_{n \in \mathbb{N}}$ then the t.f.a.e*

1. *There exists a function $f : \mathbb{R}^n \to \mathbb{R}^k$ such that $X = f(Y)$*
2. *X takes a constant value x_n on each set of the form $\{Y = y_n\}, n \in \mathbb{N}$*

Definition 80. *A family $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ of σ-algebras is called a filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t, s, t \in \mathbb{T}$*

Definition 81. *The stochastic process $X = (X_t)_{t \in \mathbb{T}}$ is said to be adapted to the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ (or $\mathcal{F}$-adapted) if X_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{T}$*
The natural filtration generated by a process X is the smallest filtration $\mathcal{F}^X$ to which X is adapted i.e. $\mathcal{F}_t^X = \sigma((X_u)_{u \leq t, u \in \mathbb{T}})$

3.2 Self-financing portfolios

At time t own $H_t^j \in \mathbb{R}$ units of j^{th} asset with price S_t^j and K_t units of the bond with price B_t
At time t the value of the portfolio is $K_t B_t + H_t \cdot S_t$
At time $t+1$ the value of the portfolio is $K_t B_{t+1} + H_t \cdot S_{t+1}$
At $t+1$ readjust the portfolio to K_{t+1} units of the bond and H_{t+1} units of the stock, between times $t+1, t+2$
Assume that these values are equal, i.e. the portfolio is self-financing

$$K_t B_{t+1} + H_t \cdot S_{t+1} = K_{t+1} B_{t+1} + H_{t+1} \cdot S_{t+1}, \quad t \in \mathbb{T}, t < T$$

Need to choose the values of variables $K_t, H_t^1, \ldots, H_t^m$ for each $t \in \mathbb{T}, t < T$ not all these random variables are free ones - they need to satisfy the T constraints of self-financing.
Can determine K_{t+1} via induction from K_t given $B > 0$ (see notes for clarification)

$$K_{t+1} = \frac{1}{B_{t+1}} \left(K_t B_{t+1} + (H_t - H_{t+1}) \cdot S_{t+1} \right)$$

We have

$$V_{t_{i+1}} - V_{t_i} = K_{t_i}(B_{t_{i+1}} - B_{t_i}) + H_{t_i}(S_{t_{i+1}} - S_{t_i})$$
$$V_{t_k} - V_0 = \sum_{i=0}^{k-1} H_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i} - H_{t_i} S_{t_i}) r$$

Which gives a simpler discounted version

$$\overline{V}_t^{x,H} = x + (H \cdot \overline{S})_t, \quad t \in \mathbb{T} \tag{66}$$

where we define the 'discrete-time stochastic integral' of H with respect to Y by

$$(H \cdot Y)_t = \sum_{s=0}^{t-1} H_s (Y_{s+1} - Y_s), \quad t \in \mathbb{T}$$

If we need to calculate K_t can do so using $\overline{V}_t$

$$K_t = \frac{1}{B_t}(V_t - H_t \cdot S_t) = \overline{V}_t - H_t \cdot \overline{S}_t$$

3.3 Arbitrage and arbitrage-free prices

Definition 82. *An adapted process $(P_t)_{t \in \mathbb{T}}$ is an Arbitrage Free Price for the derivative with payoff X_T at maturity T in the arbitrage-free market model $(B_t, S_t)_{t \in \mathbb{T}}$ if $P_t = X_T$ and the enlarged market $(B_t, S_t, P_t)_{t \in \mathbb{T}}$ is arbitrage-free.*

Theorem 83. *In the linear multi-period market model $(B_t, S_t)_{t \in \mathbb{T}}$ in eq. (66), the Domination principle holds $\iff$ there exists no arbitrage*

Theorem 84. *There exists an arbitrage in the multi-period model $(B_t, S_t)_{t \in \mathbb{T}}$ iff there exists a $s \in \mathbb{T}, s < T$ s.t there exists an arbitrage in the one-period model $(B_t, S_t)_{t = s, s+1}$*

3.4 The multi-period binomial model

Define the N-period binomial model on the following filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P}_p)$

1. $\Omega = \Omega_N = \{\omega = (\omega_1, \ldots, \omega_N) : \omega_i \in \{H, T\}\}$
2. $\mathcal{A} = \mathcal{P}(\Omega)$ all subsets of Ω
3. $\mathcal{F} = (\mathcal{F}_n)_{n=0}^N = \sigma((X_k)_{k<n})$

4. $\mathbb{P}_p$ is the probability measure on $(\Omega, \mathcal{A})$ defined by

$$\mathbb{P}_p(\{\omega\}) = p^k(1-p)^{N-k}, \quad \omega = (\omega_1, \dots, \omega_N) \in \Omega, \quad k = \sum_{i=1}^{N} \mathbb{I}_{\{\omega_i = H\}}$$

for $p \in (0,1)$

Notice that a process Y is adapted if, for each n, Y_n a function of $X_n := (X_k)_{k \leq n}$
Or equivalently if $Y_n(\omega)$ depends *only* on $\omega(n) = (\omega_1, \dots, \omega_n)$ does not depend on $(; w_{n+1}, \dots, w_N)$

On $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P}_p)$ we build the binomial market model (B, S) by asking that

- The bond price process B given by

$$B_n = B_0 \prod_{k=0}^{n-1} (1 + R_k)$$

for each $n \in \mathbb{T}$, where the interest rate process $R > -1$ is $\mathcal{F}$-adapted

- The price of underlying given by

$$S_{n+1}(\omega) = \begin{cases} (S_n U_n)(\omega), & \text{if } X_{n+1}(\omega) = H; \\ (S_n D_n)(\omega), & \text{if } X_{n+1}(\omega) = T. \end{cases}$$

where the up and down factors U_n, D_n are $\mathcal{F}$-adapted processes s.t. $0 < D < U$

Theorem 85. *The multi-period binomial model is arbitrage free* $\iff D < 1 + R < U$

3.6 Derivatives paying a cashflow

Consider derivatives that provide payoff P_t at any time $t = 0, 1, \dots, T$ where $T \in \mathbb{N}$ the expiry.
Process $P = (P_t)_{t=0,\dots,T}$ called the *cashflow* of derivative
Require it to be adapted to the filtration $\mathcal{F}$ of the market model (B, S)

Can see a derivative with cashflow P seen as the sum over $n = 0, \dots, N$ of derivatives with payoff P_n at expiry n
Denote P_k^n value at time $k \leq n$ of derivative which only has payoff P_n at expiry n
And take H_k^n the number of shares one should hold at time k to replicate it.
Value of P_k^n only defined for $k \leq n$, since derivative has expiry n, and analogously H_k^n is defined only for $k \leq n-1$

$$H_k = \sum_{n=k+1}^{N} H_k^n, \quad V_k = \sum_{n=k}^{N} D_k$$

3.7 Derivatives with random maturity

Definition 89. *A random variable with values in* $I \cup \{\infty\}$ *is called a random time.*

Definition 90. *A random time* τ *is called a stopping time if* $\{\tau \leq t\}$ *is* $\mathcal{F}_t$*-measurable for all* $t \in I$

3.8 Chooser options

American call option - gives right to exercise option and receive payment $(S_t - K)^+$, or instead wait for a later more preferable time.
Bermudan call option - right to choose stopping time τ at which receive payoff $(S_\tau - K)^+$, but only among those τ which take values in set D of possible exercise dates

3.9 American Options

Consider American option, which has payoff I_τ at time τ where τ is a stopping time, $\tau \leq T$ is chosen by the buyer.
I the intrinsic value of the derivative, τ the exercise date, and T the expiry of the option.
The buyer can exercise the option at any time $\tau \leq N$, only choosing $\tau(\omega) = t \leq N$ if $I_t(\omega) \geq 0$
If $I_t(\omega) < 0 \forall t$, he will choose $\tau = \infty$ and the option will expire worthless without being exercised, $I_\infty = 0$

3.10 Conditional probability and conditional Expectation

Theorem 97. $\mathbb{P}(\cdot \mid B) : \mathcal{A} \to [0,1], \quad \mathbb{P}(A \mid B) := \mathbb{P}(A \cap B)/\mathbb{P}(B), \quad$ *for* $A \in \mathcal{A}$
$\mathbb{P}(\cdot \mid B)$ *a probability on* $\mathcal{A}$, *and*

$$\mathbb{E}^{\mathbb{P}(\cdot|B)}[X] = \mathbb{E}^{\mathbb{P}}[X 1_B]/\mathbb{P}(B)$$

for any random variable $X \geq 0$, *and thus for all* X *s.t.* $\mathbb{E}^{\mathbb{P}}[|X| 1_B] < \infty$

Definition 98. *An atom of a* σ*-algebra* $\mathcal{F}$ *is a non-empty* $A \in \mathcal{F}$ *such that* $B \subseteq A, B \in \mathcal{F}$ *imply that either* $B = \emptyset$ *or* $B = A$
The family of atoms of $\mathcal{F}$ *is denoted by* $\mathcal{A}(\mathcal{F})$

Theorem 99. *Assume* $\mathcal{F}$ *is a finite* σ*-algebra on* Ω, *and denote with* $A_{\mathcal{F}}(\omega)$ *the intersection of all the* $A \in \mathcal{F}$ *which contains* $\omega \in \Omega$
Then $A_{\mathcal{F}}$ *is the smallest* $A \in \mathcal{F}$ *which contains* ω. *The family* $\mathcal{A}(\mathcal{F})$ *of atoms of* $\mathcal{F}$ *is a finite partition of* Ω, *and*

$$\mathcal{A}(\mathcal{F}) = \{A_{\mathcal{F}}(\omega) : \omega \in \Omega\}$$

so $A_{\mathcal{F}}(\omega)$ *is the (only) atom of* $\mathcal{F}$ *containing* $\omega \in \Omega$. *The function* $\Pi to \sigma(\Pi)$, *mapping finite partitions of* Ω *to finite* σ*-algebras on* Ω, *is a bijection and has inverse* $\{ \mapsto \mathcal{A}(\mathcal{F})$; *in particular, if* $\mathcal{F} = \sigma(X)$ *for some random variable* X, *which satisfies the below eq. (79), then* $\mathcal{A}(\mathcal{F}) = \{X = x_k\}_{k=1}^n$

$$P(\{X = x_k\}) > 0 \quad \forall k = 1, \dots, n, \quad \mathbb{P}\left(X \notin \bigcup_{k=1}^n \{x_k\} \right) = 0 \tag{79}$$

3.11 Properties of the conditional expectation

Theorem 100. *For all* $X \in L^1$ *and* σ*-algebra* $\mathcal{G} \subseteq \mathcal{F}$ *the net* $(\mathbb{E}[X \mid \mathcal{H}])_{\mathcal{H} \in \mathbb{H}(\mathcal{G})}$ *converges in* L^1.
Its L^1 *limit is called the conditional expectation of* X *given* $\mathcal{G}$, *denoted with* $\mathbb{E}[X \mid \mathcal{G}]$

Theorem 101. $Z := \mathbb{E}[X \mid \mathcal{G}]$ *is* $\mathcal{G}$*-measurable and satisfies*

$$\mathbb{E}[ZW] = \mathbb{E}[XW] \textit{ for all } \mathcal{G} - \textit{measurable and bounded } W; \tag{83}$$

moreover, $Z = \mathbb{E}[X \mid \mathcal{G}]$ *is the unique* $\mathcal{G}$*-measurable random variable* $Z \in L^1$ *which satisfies eq. (83)*

Properties of conditional expectation:

1. *Linearity:* $\mathbb{E}(X + Z \mid \mathcal{G}) = \mathbb{E}[X \mid \mathcal{G}] + \mathbb{E}[Z \mid \mathcal{G}]$
2. *Independence:* If X is independent of $\mathcal{G}$, then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$
   In particular, any constant $c \in \mathbb{R}$ satisfies $\mathbb{E}(c \mid \mathcal{G}) = c$, and if $\mathcal{G}$ is the trivial σ-algebra $\{\varnothing, \Omega\}$ then any X satisfies $\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}(X)$
3. *Taking out what is known:* if X is $\mathcal{G}$-measurable then $\mathbb{E}(XZ \mid \mathcal{G}) = X\mathbb{E}(Z \mid \mathcal{G})$, and in particular $\mathbb{E}(X \mid \mathcal{G}) = X$
4. *Iterated conditioning:* if $\mathcal{H} \subseteq \mathcal{A}$ a σ-algebra and $\mathcal{G} \subseteq \mathcal{H}$ then $\mathbb{E}(\mathbb{E}(X \mid \mathcal{H}) \mid \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G})$; in particular $\mathbb{E}[\mathbb{E}(X \mid \mathcal{H})] = \mathbb{E}[X]$
5. *Jensen inequality:* if $\phi : \mathbb{R} \to \mathbb{R}$ is convex then $\mathbb{E}[\phi(X) \mid \mathcal{G}] \geq \phi\mathbb{E}[X \mid \mathcal{G}]$

Theorem 102. *Assume* $X \in L^2(\mathcal{A})$*. Then* $\mathbb{E}[X \mid \mathcal{G}]$ *is the unique* C *minimiser of* $\mathbb{E}[(X-C)^2]$ *across* $C \in L^2(\mathcal{A})$ *.*
Equivalently $\mathbb{E}[X \mid \mathcal{G}]$ *is the unique* $C \in L^2(\mathcal{A})$ *s.t.*

$$\mathbb{E}[(X-C)W] = 0 \text{ for all } W \in L^2(\mathcal{G})$$

Lemma 103. *The set* $L^2 = L^2(\mathbb{P})$ *of random variables* X *s.t.* $\mathbb{E}(X^2) < \infty$ *is a vector space, and if* $X, Y \in L^2(\mathbb{P})$ *then* $XY \in L^1(\mathbb{P})$

3.12 The RNPF in the multi-period binomial model

Compute up and down factors

$$U_n(\omega) := \frac{S_{n+1}((\omega(n), H))}{S_n(\omega(n))}, \quad D_n(\omega) := \frac{S_{n+1}((\omega(n), T))}{S_n(\omega(n))}$$

and define risk-neutral transition-probabilities $\widetilde{P}_n$ and $\widetilde{Q}_n = 1 - \widetilde{P}_n$ by asking that

$$\overline{S}_n(\omega(n)) = \widetilde{P}_n(\omega(n))\overline{S}_{n+1}((\omega(n), H)) + \widetilde{Q}_n(\omega(n))\overline{S}_{n+1}((\omega(n), T))$$

Solving to get

$$\widetilde{P}_n(\omega) = \widetilde{P}_n(\omega(n)) := \frac{(1+R_n) - D_n}{U_n - D_n}(\omega(n)), \quad n = 0, \ldots, N-1 \tag{86}$$

$$\widetilde{Q}_n(\omega) = \widetilde{Q}_n(\omega(n)) := \frac{U_n - (1+R_n)}{U_n - D_n}(\omega(n)), \quad n = 0, \ldots, N-1 \tag{87}$$

Compute $V_n = V_n^{x,G}$

$$\overline{V}_n(\omega(n)) = \widetilde{P}_n(\omega(n))\overline{V}_{n+1}((\omega(n), H)) + \widetilde{Q}_n(\omega(n))\overline{V}_{n+1}((\omega(n), T)) \tag{88}$$

$$\overline{V}_n(\omega(n)) = \mathbb{E}^{\mathbb{Q}|\{X(n)=\omega(n)\}}[\overline{V}_{n+1}] \tag{90}$$

$$\overline{V}_n = \mathbb{E}^{\mathbb{Q}}[\overline{V}_{n+1} \mid \mathcal{F}_n] \tag{91}$$

Lemma 105. *The map* $\mathbb{Q} \mapsto \widetilde{P}$ *given by equation (89) is a bijection between*

1. *probabilities* $\mathbb{Q}$ *on* $(\Omega, \mathcal{A})$
2. $\mathcal{F}$*-adapted processes* $\widetilde{P} = (\widetilde{P})_{n \leq N}$ *on* $(\Omega, \mathcal{A})$ *with values in* $[0,1]$

Moreover $\mathbb{Q} \sim \mathbb{P} \iff O < \widetilde{P} < 1$

3.13 The FTAP in the multi-period setting

As $\widetilde{P}$ is determined by eq. (85) which is eq. (88) with $V = S$, eq. (91) states that $\mathbb{Q}$ satisfies

$$\overline{S}_n = \mathbb{E}^{\mathbb{Q}}[\overline{S}_{n+1} \mid \mathcal{F}_n]$$

Definition 107.
1. *An adapted process* $Y = (Y_t)_{t \in \mathbb{T}}$ *a martingale if* $Y_t \in L^1(\mathbb{P})$ *and* $\mathbb{E}[Y_t \mid \mathcal{F}_s]$ *for each* $s \leq t, s, t \in \mathbb{T}$
2. *A probability* $\mathbb{Q}$ *on* $\mathcal{F}$ *is a Martingale Measure if* $\overline{S}$ *is a martingale on* $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{Q})$
Say $\overline{S}$ *a* $\mathbb{Q}$*-martingale*
Such $\mathbb{Q}$ *is an EMM if* $\mathbb{Q} \sim \mathbb{P}$*. The set of EMM denoted by* $\mathcal{M}(\overline{S})$

Theorem 108 (1st FTAP)**.** *A multi-period market* $(B_t, S_t)_{t \in \mathbb{T}}$ *is arbitrage free* $\iff \mathcal{M}(\overline{S}) \neq \varnothing$

Corollary 109. *Let* $(B_t, S_t)_{t \in \mathbb{T}}$ *be a multi-period market free of arbitrage. Then* (B, S) *is complete* $\iff$ *the EMM is unique (i,e.* $\mathcal{M}(\overline{S})$ *is a singleton)*

3.14 Permutation-invariant processes and recombinant trees

To find AFP $Y := (Y_n)_{n<N}$ for binomial model - work by backward induction. Setting $\overline{V}_n = \overline{V}_n^{x,G} := \overline{Y}_n$ and using the RNPF eq. (91) to compute $\overline{V}_n = \mathbb{E}^{\mathbb{Q}}(\overline{V}_{n+1} \mid \mathcal{F}_n)$ and G_n for each $0 \leq n \leq N-1$

Problem: amount of computation grows exponentially with n - as its proportional to the # of paths $(\omega_1, \ldots, : w_n) = \#\{H, T\}^n = 2^n$

Solution - consider for (B, S) only those adapted processes W such that for each n , W_n takes the same value at the point $(\omega_1, \ldots, \omega_n)$ as at the point $\big(\sigma(\omega_1), \ldots, \sigma(\omega_n)\big)$, where σ any permutation of $\Omega_n = \{H, T\}^n$
Call such W *permutation-invariant*
If W is permutation-invariant W_n takes at most $n+1$ possible values, as it is a function only of the number $k = 0, \ldots, n$ of coin tosses which results in Heads. As the number of values of W_n grows linearly in n - instead of exponentially - the amount of computation is reduced.

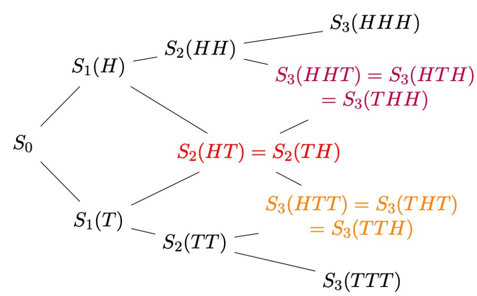

Figure 1: Recombinant tree

Suppose (B, S) is permutation invariant. If the value $\overline{Y}_n$ of a derivative at time N depended just on (B_N, S_N), i.e. $\overline{Y}_n = f_N(B_N, S_N)$, then we would only need to keep track of the $N+1$ values of (B_N, S_N) to compute $\overline{Y}_n$

For a general derivative we have $\overline{Y}_n = f_N((B_k, S_k)_{k \leq N})$, we would still only need to keep track of $\sum_{k=0}^{n}(k+1) = \frac{1}{2}(N+1)(N+2) \sim N^2$ values - which is much less than 2^N

3.15 Independence

Definition 110. *Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we say two events $A, B \in \mathcal{A}$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$*

Definition 111. *Given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ say finitely many events $F_i \in \mathcal{A}, i \in J = \{i_j\}_{j=1}^{n}$ are $\mathbb{P}$-independent if we have*

$$\mathbb{P}(G_{i_1} \cap \ldots \cap G_{i_n}) = \prod_{j=1}^{n} \mathbb{P}(G_{i_j})$$

For any $G_{i_j} \in \{F_{i_j}, F_{i_j}^c\}, j = 1, \ldots, n$

Definition 113. *Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an arbitrary collection of events $G_i \in \mathcal{A}, i \in I$ are independent if $\{G_i\}_{i \in J}$ are independent for every finite $J \subseteq I$*
Arbitrary collection $\{\mathcal{G}_i\}_{i \ni I}$ of sub-σ-algebras of $\mathcal{A}$ are independent if for every choice of sets $G_i \in \mathcal{G}_i, i \in I$, the sets $\{G_i\}_{i \in I}$ are independent
An arbitrary collection of random vectors $X_i : \Omega \to \mathbb{R}^{k_i}, i \in I$ are independent if the σ-algebras $\{\sigma(X_i)\}_{i \in I}$ are independent

Theorem 114. *Given random vectors $X_j : \Omega \to \mathbb{R}^{k_j}, j = 1, \ldots, n$ the following are equivalent:*

1. *$(X_j)_j$ are independent*

2.

$$\mathbb{E}\left[\prod_{k=1}^{n} f_j(X_j)\right] = \prod_{k=1}^{n} \mathbb{E}[f_j(X_j)] \tag{97}$$

holds for any bounded and Borel functions $f_j : \mathbb{R}^{k_j} \to \mathbb{C}, j = 1, \ldots, n$ of the form $f_j(x) = \exp(i \cdot t_j \cdot x), t_j \in \mathbb{R}^{k_j}$

3. eq. (97) holds for any functions $f_j : \mathbb{R}^{k_j} \to \mathbb{C}, j = 1, \ldots, n$ of the form $f_j(x) = \exp(i \cdot t_j \cdot x), t_j \in \mathbb{R}^{k_j}$

Remark 115. *1. if X_j has values in $\mathbb{R}^+$ for each j and eq. (97) holds for every f_j of the form $f_j(x) = \exp(t_j x), t_j \in \mathbb{R}$, or*

2. if each X_j has values in a bounded set B_j ($\mathbb{P}(X_j \notin B_j) = 0$) and eq. (97) holds for every f_j which is a polynomial

Then the $(X_j)_j$ are independent

Remark 116. *For $\mathbb{T} = \{0, 1, \ldots, N\}$ or $\mathbb{T} = \mathbb{N}$*

1. *$(X_i)_{i \in \mathbb{T}}$ are independent*
2. *For every $J, K \subseteq \mathbb{T}$ s.t. $J \cap K = \varnothing$ the two random vectors $Y := (X_i)_{i \in J}$ and $Z := (X_i)_{i \in K}$ are independent*
3. *For every $k \in \mathbb{T} \setminus \{0\}$, the two random vectors $X(k-1) := (X_i)_{i=0}^{k-1}$ and X_k are independent*

Theorem 117. *Given two σ-algebras $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, the following are equivalent:*

1. *$\mathcal{B}, \mathcal{C}$ are $\mathbb{P}$-independent*
2. *for all $B \in \mathcal{B}$ the random variable $\mathbb{P}(B \mid \mathcal{C})$ is constant*
3. *for every $\mathcal{B}$-measurable $Y \in L^1(\mathbb{P})$, the random variable $\mathbb{E}(Y \mid \mathcal{C})$is constant*

and in this case $\mathbb{P}(B \mid \mathcal{C}) = \mathbb{P}(B)$, and $\mathbb{E}[Y \mid \mathcal{C}] = \mathbb{E}[Y]$

Theorem 118. *If two random variables X, Y take finitely many values $\{x_i\}_{i=1}^n, \{y_j\}_{j=1}^m$ then X, Y independent iff, for each i, j the two sets $\{X = x_i\}, \{Y = y_i\}$ are independent*

Remark 119. *If $\mathcal{B}$ finite, if $\mathbb{P}(B \mid \mathcal{C}) = \mathbb{P}(B)$ holds for every atom of B of $\mathcal{B}$ then it holds for every finite union of atoms, i.e. for every $B \in \mathcal{B}$; moreover if $\mathbb{P}(B \mid \mathcal{C}) = \mathbb{P}(B)$ then*

$$\mathbb{P}(B^c \mid C) = 1 - \mathbb{P}(B \mid \mathcal{C}) = 1 - \mathbb{P}(B) = \mathbb{P}(B^c)$$

Thus, a random variable X which only takes 2 values x_1, x_2 is independent of a σ-algebra $\mathcal{G}$ iff $\mathbb{P}(X = x_1 \mid \mathcal{G})$ is constant

Remark 120. *Combining remarks 116 + 119.*
Follows that random variables $(X_i)_{i=1}^N$ with values in $\{H, T\}$ are $\mathbb{P}$-independent iff, for every $k \in \{1, \ldots, N-1\}$

$$\mathbb{P}(X_{k+1} = H \mid \sigma\left(X_1, \ldots, X_n\right)) \quad \text{is constant}$$

Remark 121. *WARNING: if X, Y, Z are pairwise independent, then not necessarily X, Y, Z are independent*

3.16 Pricing and hedging fast using Markov Processes

Definition 122. *An adapted process* $X = (X_t)_{t\in\mathbb{T}}$ *on a filtered prob. space* $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ *is Markov if*

$$\forall f \text{ Borel-measurable }, \forall s \leq t, s, t \in \mathbb{T}, \quad \mathbb{E}(f(X_t) \mid \mathcal{F}_s) = \mathbb{E}(f(X_t) \mid X_s)$$

Lemma 124 (Independence Lemma)**.** *If* X *a* $\mathbb{R}^k$*-valued, and* Y *a* $\mathbb{R}^n$*-valued random vector on* $(\Omega, \mathcal{A}, \mathbb{P}), \mathcal{G} \subset \mathcal{A}$ *a* σ*-algebra,*X *is* $\mathcal{G}$*-measurable and* Y *is independent of* $\mathcal{G}, f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ *a Borel function, then*

$$\mathbb{E}(f(X, Y) \mid \mathcal{G}) = g(x) \quad \text{for} \quad g(x) = \mathbb{E}(f(x, Y)), \quad g : \mathbb{R}^k \to \mathbb{R}$$

Corollary 126. *If a* $\mathcal{F}$*-adapted process* W *satisfies* $W_{k+1} = f_k(W_k, X_{k+1})$ *for each* $k \in \mathbb{Z}$*, where* f_k *is some Borel function and* X_{k+1} *is independent of* $\mathcal{F}_k$*, then* W *is Markov and* $\mathbb{E}(f(W_{k+1} \mid \mathcal{F}_k)) = g(W_k)$ *for all* $k \in \mathbb{Z}$*, where*

$$g(\omega) := \mathbb{E}[f(f(\omega, X_{k+1}))]$$

4 Continuous Time models

Lol

NON EXAMINABLE MATERIAL