1 Problem sheet 1: Discrete-time Markov chains

1.1 Prerequisites: Lecture 2

Exercise 1- 1: Show by induction that for a discrete Markov chain $(X_n)_{n\in\mathbb{N}_0}$, we have

$$\mathrm{P}(X_{n+m} = x_{n+m}|X_n = x_n, \ldots, X_0 = x_0) = \mathrm{P}(X_{n+m} = x_{n+m}|X_n = x_n),$$

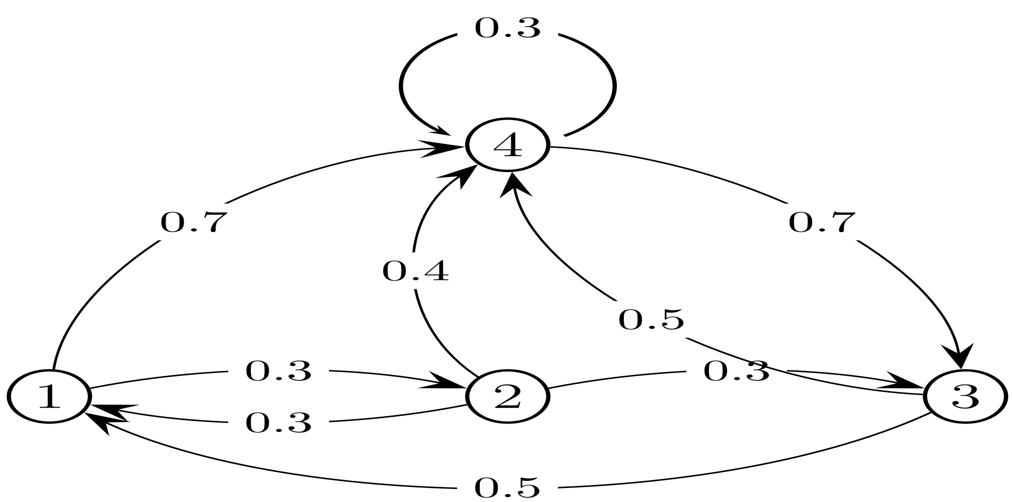
for $m \in \mathbb{N}$ and for all $x_{n+m}, x_n, \ldots, x_0 \in E$.

Exercise 1- 2: Show that a $K \times K$-dimensional *stochastic matrix* has at least one eigenvalue equal to 1. Hence show that if $\mathbf{P}$ is a stochastic matrix, then so is $\mathbf{P}^n$ for all $n \in \mathbb{N}$.

Exercise 1- 3: For each matrix, decide whether it is stochastic. If it is, draw the corresponding transition diagram (assuming the state space is given by $E = \{1, 2, 3\}$).

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0.5 & -0.5 & -1 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$ (b) $\begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0 & 0.7 & 0.3 \\ 0.1 & 0.9 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.9 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$

Exercise 1- 4: Derive the transition matrix from the following transition diagram.



Exercise 1- 5: Suppose that the movements of a particle are recorded at discrete times on a discrete lattice $\{0, 1, \ldots, a\}$, such that if it hits either 0 or a it never leaves. Given that the particle is at the points $\{1, \ldots, a-1\}$ it may jump up the lattice with probability p and down with probability $1 - p$. What are the transition probabilities for this Markov chain?

Exercise 1- 6: Let X_n be the maximum reading obtained in the first n rolls of a fair die (for $n \in \mathbb{N}$). Show that $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain, and give the transition probabilities.

Exercise 1- 7: Consider the transition matrix of a Markov chain with state space $E = \{1, 2, 3\}$ given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

Derive a general formula $p_{12}(n)$ for $n \in \{0, 1, 2, \ldots\}$. *Hint:* Compute the eigenvalues of $\mathbf{P}$ and diagonalise $\mathbf{P}$ to compute $\mathbf{P}^n$!

1.2 Prerequisites: Lecture 3

Exercise 1- 8: Show that a discrete random process $(X_n)_{0\leq n\leq N}$ $(N < \infty)$ is a discrete-time, time-homogeneous Markov chain on the state space E if and only if

$$\mathrm{P}(X_0 = x_0, X_1 = x_1, \ldots, X_N = x_N) = \mathrm{P}(X_0 = x_0)p_{x_0x_1}p_{x_1x_2}\cdots p_{x_{N-1}x_N},$$

for all $x_0, x_1, \ldots, x_N \in E$.

Exercise 1- 9: Using the notation from the lecture notes, show by induction that the marginal distribution of a Markov chain satisfies

$$\boldsymbol{\nu}^{(n)} = \boldsymbol{\nu}^{(0)}\mathbf{P}_n,$$

for $n \in \mathbb{N}$.

1.3 Prerequisites: Lecture 4

Exercise 1- 10: Suppose that $p, q \in (0, 1)$ and $p + q = 1$. The transition matrix of a Markov chain with state space $E = \{0, 1, \ldots, \}$ is:

$$\mathbf{P} = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ q & 0 & 0 & p & 0 & \cdots \\ q & 0 & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(a) Find the first return probabilities for state 0, i.e. find $f_{00}(n)$ for all $n \in \mathbb{N}$.

(b) Use your results from (a) to find f_{00}. Is state 0 recurrent?

Exercise 1- 11: (From the exam paper in 2019-2020) Suppose the influenza virus exists in K different strains, where $K \geq 2$. Each year, the virus either stays the same with probability $1 - a$, for $a \in (0, 1)$, or mutates to any of the other strains with equal probability. Suppose you can model the virus mutation by a discrete-time homogeneous Markov chain.

1. We denote the state space by $E = \{1, \ldots, K\}$. State the corresponding 1-step transition probabilities of the Markov chain.
2. You decide to group the states: You consider the modified state space $\widetilde{E} = \{I, O\}$ where I stands for the initial state and O for the collection of the other $K - 1$ states.
   (a) State the corresponding 1-step transition probabilities of the Markov chain on $\widetilde{E}$.
   (b) Show that, for $n \in \mathbb{N}$,
   $$p_{II}(n + 1) = p_{II}(n)\left\{1 - a - \frac{a}{K - 1}\right\} + \frac{a}{K - 1},$$
   and state all results from the lectures which you apply in your proof.

1.4 Prerequisites: Lecture 6

Exercise 1- 12: Let $\alpha, \beta \in (0, 1)$. We are given a Markov chain on a state space $E = \{0, 1\}$, with transition matrix

$$\begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}.$$

(a) Derive the generating function associated with the sequence $(f_{00}(n))_{n\in\mathbb{N}_0}$. *H*int: Find $f_{00}(n)$ for all $n \in N_0$ for this Markov chain. Then define the corresponding generating function as $G(s) = \sum_{n=0}^{\infty} f_{00}(n)s^n$ for $|s| < 1$.

(b) Find, using generating functions, the expected return time to state 0. I.e. compute $\mu_0 = \frac{d}{ds}G(s)\big|_{s=1} = \sum_{n=1} nf_{00}(n)$.

(c) What is the "holding time" (i.e. the time spent in a particular state) distribution for state 0 that you have derived in this question?

1.5 Prerequisites: Lecture 6

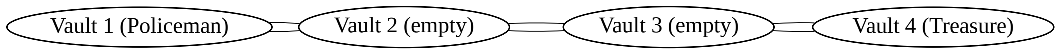
Exercise 1- 13: The following question is adapted from Grimmett & Stirzaker (2001*b*,*a*) Problem 6.15.6: Let $i, j \in E$ and suppose that $i \leftrightarrow j$.

Show that there is positive probability of reaching j from i without revisiting i in the meantime. Deduce that, if the chain is irreducible and recurrent, then $f_{ij} = 1, \forall i, j \in E$.

2 Problem sheet 2: Discrete-time Markov chains

2.1 Prerequisites: Lecture 7

Exercise 2- 14: Consider a sequence of 4 vaults labelled as $E = \{1, 2, 3, 4\}$. In vault 1 there is a policeman and in vault 4 there is a treasure chest. Vaults 2 and 3 are empty, see the picture below.



Suppose there is a thief walking through the vaults and you can model the location of the thief, i.e. the corresponding vault number, at each point in time by a homogeneous Markov chain denoted by $(X_n)_{n\in\{0,1,2,\dots\}}$.

If the thief is in vault 1 (together with the policeman), he will run out of vault 1 to vault 2 with probability one. If the thief is in vault 4 (with the treasure), he will stay there forever. If the thief is in vault 2 or 3, he will either go left with probability 1/2 or he will go right with probability 1/2. Moreover, assume that the thief is not very clever, so he might return to vault 1 (with the policeman) several times. Also, suppose the policeman never manages to catch the thief and jail him (even if they are in the same vault).

(a) State the transition probabilities for this Markov chain.

(b) Suppose the thief starts his journey in vault 1. What is the expected number of moves required until the thief reaches the treasure chest? Justify your answer carefully.

2.2 Prerequisites: Lecture 9

Exercise 2- 15: A transition matrix is called *doubly stochastic* if all its column sums equal 1, that is, if $\sum_i p_{ij} = 1$ for all $i, j \in E$.

(a) Assume the Markov chain has finite state space, i.e. $|E| = K < \infty$.

- Show that if the transition matrix is doubly stochastic, then all states are positive recurrent.
- Show that if the transition matrix is doubly stochastic and, in addition, if the chain is irreducible and aperiodic, then $p_{ij}(n) \to \frac{1}{K}$ as $n \to \infty$.

(b) Assume the Markov chain has infinite state space, i.e. $|E| = \infty$.

- Show that if the chain is irreducible and the transition matrix is doubly stochastic, then all states are either null recurrent or transient.

Exercise 2- 16: The following question is adapted from Grimmett & Stirzaker (2001*b*,*a*) Problem 6.15.7:

Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a recurrent irreducible Markov chain on the state space E with transition matrix $\mathbf{P}$, and let $\mathbf{x}$ be a positive solution of the equation $\mathbf{x} = \mathbf{x}\mathbf{P}$.

(a) Show that

$$q_{ij}(n) = \frac{x_j}{x_i} p_{ji}(n), \qquad i, j \in E, n \in \mathbb{N},$$

defines the n-step transition probabilities of a recurrent irreducible Markov chain on E whose first-passage probabilities are given by

$$g_{ij}(n) = \frac{x_j}{x_i} l_{ji}(n), \qquad i \neq j, n \in \mathbb{N}, \tag{2.1}$$

where $l_{ji}(n) = \mathrm{P}(X_n = i, T_j \geq n | X_0 = j)$ and $T_j = \min\{m \in \mathbb{N} : X_m = j\}$.

(b) Show that $\mathbf{x}$ is unique up to a multiplicative constant.

Exercise 2- 17: Let T be a nonnegative integer-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and let $A \in \mathcal{F}$ be an event with $\mathrm{P}(A) > 0$. Show that

$$\mathrm{E}(T|A) = \sum_{n=1}^{\infty} \mathrm{P}(T \geq n|A).$$

2.3 Prerequisites: Lecture 10

Exercise 2- 18: We consider Markov chains with state space $E = \{0, 1, 2, 3\}$. For each of the Markov transition matrices below:

$$(a) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (b) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix} \quad (c) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

- Specify the communicating classes and determine whether they are transient or recurrent;
- Decide whether or not they have a unique stationary distributions;
- Find a stationary distribution for each of them and show that it is not unique where appropriate.

Exercise 2- 19: Consider a discrete-time homogeneous Markov chain $(X_n)_{n \in \mathbb{N}_0}$ with state space $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0.75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Draw the transition diagram.

(b) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent. *Please note that you need to justify your answers.*

(c) Find all stationary distributions.

(d) For each communication class, pick a state i and find the first passage probabilities $f_{ii}(n) = \mathrm{P}(X_n = i, X_{n-1} \neq i, \dots, X_1 \neq i | X_0 = i)$ for all $n \in \mathbb{N}$ and derive $f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n)$.

Exercise 2- 20: Suppose we have a Markov chain with finite state space E, i.e. $K = |E| < \infty$, and transition matrix $\mathbf{P}$. Suppose for some $i \in E$ that

$$p_{ij}(n) \to \pi_j \text{ as } n \to \infty \text{ for all } j \in E.$$

Then $\boldsymbol{\pi}$ is a stationary distribution.

Exercise 2- 21: Consider a discrete-time homogeneous Markov chain $(X_n)_{n \in \mathbb{N}_0}$ with state space $E = \{1, 2\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

(a) Derive $\lim_{n \to \infty} \mathrm{P}(X_n = i)$ for $i \in \{1, 2\}$.

(b) Find

$$\lim_{N\to\infty} \frac{1}{N+1} \mathrm{E}\left(\sum_{n=0}^{N} e^{X_n}\right).$$

Hint: You may use the following result from Analysis without a proof: Let $(x_n)_{n\in\mathbb{N}_0}$ *be a real-valued convergent sequence with* $\lim_{n\to\infty} x_n = x$*. Then* $\lim_{m\to\infty} \frac{1}{m+1}\sum_{n=0}^{m} x_n = x$*.*

3 Problem sheet 3: Exponential distribution and Poisson processes

3.1 Prerequisites: Lecture 12

Exercise 3- 22: Prove Theorem 4.1.4: Let $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n > 0$. Consider independent random variables $H_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \ldots, n$. Let $H := \min\{H_1, \ldots, H_n\}$. Then

(a) $H \sim \text{Exp}(\sum_{i=1}^n \lambda_i)$.

(b) For any $k = 1, \ldots, n$, $\text{P}(H = H_k) = \lambda_k / (\sum_{i=1}^n \lambda_i)$.

Exercise 3- 23: A lump of radioactive material has 10^{26} atoms. Each atom decays after an exponential time with mean 10^{26} seconds, independently of the decay of other atoms.

(a) Let X_1 be the time taken for the first decay to occur. Why is $X_1 \sim \text{Exp}(1)$? Let X_2 be the time between the first decay and the second. What is the distribution of X_2?

(b) What is the probability a given atom decays in the first minute? Let N be the total number of decays in the first minute. By noting that N has a binomial distribution, show that N has an approximate Poisson distribution with mean 60.

3.2 Prerequisites: Lecture 15

Exercise 3- 24: Let $X_1, \ldots, X_n$ be i.i.d. random variables, following the uniform distribution on $(0, t)$ for $t > 0$. Let $X_{(1)}, \ldots, X_{(n)}$ denote the corresponding order statistics. That is, they have joint density function

$$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)}, \ldots, x_{(n)}) = \begin{cases} \frac{n!}{t^n}, & \text{if } 0 < x_{(1)} < \cdots < x_{(n)} < t, \\ 0, & \text{otherwise.} \end{cases}$$

Let $1 \leq k \leq n$; show that

$$\text{E}[X_{(k)}] = \frac{tk}{n+1}.$$

Note that you do not have to derive the marginal density of $X_{(k)}$ from first principles.

Exercise 3- 25: Show that Definition 5.4.4 of a Poisson process implies Definition 5.3.4.

3.3 Prerequisites: Lecture 16

Exercise 3- 26: Consider two independent Poisson processes $\{N_t^{(1)}\}_{t\geq 0}$ and $\{N_t^{(2)}\}_{t\geq 0}$ (of rates $\lambda_1 > 0$ and $\lambda_2 > 0$), and we define a new stochastic process

$$N_t = N_t^{(1)} + N_t^{(2)}.$$

Show that $\{N_t\}_{t\geq 0}$ is a Poisson process of rate $\lambda_1 + \lambda_2$.

Exercise 3- 27: Customers arrive at a bank according to a Poisson process $(N_t)_{t\geq 0}$ at a mean rate of $\lambda = 10$ per minute. 60% of the customers wish to withdraw money (type A), 30% wish to pay in money (type B), and 10% wish to do something else.

(a) What is the probability that more than 5 customers arrive in 30 seconds?

(b) What is the probability that in 1 minute, 6 type A customers, 3 type B customers, and 1 type C customers arrive?

(c) If 20 customers arrive in 2 minutes, what is the probability that just one wants to carry out a type C transaction?

(d) What is the probability that the first 3 customers arriving require only to make a type A transaction?

(e) How long a time will elapse until there is a probability of 0.9 that at least one customer of type A and one of type B will have arrived? (You will need to solve this numerically).

Exercise 3- 28: Suppose that cars arrive at the petrol station according to a Poisson process, $\{N_t\}_{t\geq 0}$ of rate $\lambda > 0$. In addition, independently, a car is green with probability p; let $\{N_t^g\}_{t\geq 0}$ denote the number of green cars that have appeared. Show that $\{N_t^g\}_{t\geq 0}$ is a Poisson process.

Exercise 3- 29: A bank opens at 10.00am and customers arrive according to a non-homogeneous Poisson process at a rate $10(1+2t)$, measured in hours, starting from 10.00.

(a) What is the probability that two customers have arrived by 10.05?

(b) What is the probability that 6 customers arrive between 10.45 and 11.00?

(c) What is the probability that more than 50 customers arrive between 11.00 and 12.00?

(d) What is the median time to the first arrival after the bank opens?

(e) By what time is there a probability of 0.95 that the first customer after 11.00 will have arrived?

3.4 Prerequisites: Lecture 17

Exercise 3- 30: Let $\{N_t\}$ be a Poisson process of rate $\lambda > 0$ and $Y, Y_1, Y_2, \ldots$ be a sequence of i.i.d random variables, such that their characteristic function exists. Further $\{N_t\}$ and $\{Y_j\}$ are independent. Let

$$S_t = \sum_{j=1}^{N_t} Y_j.$$

Find the characteristic function of S_t, $t > 0$.

Exercise 3- 31: A person makes shopping expeditions according to a Poisson process with rate $\lambda > 0$. The number of purchases he makes during each expedition is distributed according to a geometric distribution $\text{Geometric}(p)$. What are the mean and variance of the total number of purchases made in time t? *Hint: Please use the following probability mass function of the geometric distribution* $p(y) = p(1-p)^{y-1}, \quad y = 1, 2 \ldots.$

Exercise 3- 32: A physicist has a large lump of radioactive material of very long half- life. She records radioactive decays from this lump using a suitable counter. The counter is switched on at 12.00 noon and then left running. Let N_t be the total number of decays recorded by the counter after it has been switched on for t hours.

(a) Let D_t be the total number of decays which occur in the radioactive material in the period of t hours starting at noon. Suppose that $\{D_t\}_{t\geq 0}$ is a Poisson process of rate $\mu > 0$. Each decay is recorded by the counter with probability p independently of whether other decays are recorded and independently of when decays occur. What is the distribution of N_t? (Hint: Consider the probability generating function).

(b) Suppose $\{N_t\}_{t\geq 0}$ is a Poisson process of rate $\lambda = 3$ per hour. Having switched the counter on at 12 noon, the physicist goes to lunch. She returns at 1pm. What is the probability that the counter has recorded exactly one decay by this time? Given that exactly one decay has occurred, show that the probability that this decay occurred after 12.45pm is 0.25.

4 Problem sheet 4: Continuous-time Markov chains

4.1 Prerequisites: Lecture 18

Exercise 4- 33: Let $N = (N_t)_{t\geq 0}$ denote a Poisson process with rate $\lambda > 0$. Define a stochastic process $Z = (Z_t)_{t\geq 0}$ with $Z_t = (-1)^{N_t}$.

(a) Determine the state space E_Z of all possible values Z can take.

(b) Sketch a sample path of the process Z and describe how long on average you need to wait until the process switches between different values in E_Z.

(c) Find the probability mass function of Z_t for $t \geq 0$.

(d) Find $\mathrm{E}(Z_t)$ for $t \geq 0$.

(e) Find $\mathrm{P}(Z_s = Z_t)$ for $0 \leq s < t$.

(f) Determine whether Z is a continuous-time Markov chain and prove/justify your answer carefully.

Exercise 4- 34: A machine can be in one of two states: working or being repaired. When it is in the "working" state it functions for a time that is exponentially distributed (parameter $\lambda > 0$) before switching to the "being repaired" state. When it is in the "being repaired" state it functions for a time that is exponentially distributed (parameter $\nu > 0$) before switching to the "working" state. We assume independence between the corresponding holding times.

(a) Given that the machine starts in the "working" state, what is the mean time until:

- it breaks down for the first time?
- it breaks down for the third time?

(b) What is the variance of the time until

- it breaks down for the first time?
- it breaks down for the third time?

Exercise 4- 35: The following question is adapted from Grimmett & Stirzaker (2001*b*,*a*), Problem 6.15.14.

Let X be a continuous-time Markov chain with countable state space E and standard semigroup $\mathbf{P}_t$.

(a) Show that $p_{ij}(t)$ is a continuous function of t. *Hint:* Use the Chapman-Kolmogorov equations.

(b) Next, let $g(t) = -\log(p_{ii}(t))$. Show that 1) g is a continuous function, 2) $g(0) = 0$, and 3) g is subadditive, i.e. $g(s+t) \leq g(s) + g(t)$ for all $s, t \geq 0$. From a result from analysis (which you do not need to prove) you may then conclude that

$$\lim_{t\downarrow 0} \frac{g(t)}{t} = \lambda \quad \text{exists and } \lambda = \sup_{t>0} \frac{g(t)}{t} \leq \infty.$$

Deduce that $g_{ii} = \lim_{t\downarrow 0} t^{-1}(p_{ii}(t) - 1)$ exists, but may be equal to ∞.

4.2 Prerequisites: Lecture 19

Exercise 4- 36: In this question we will study recurrence and transience for continuous-time Markov chains. We first introduce a definition and state an important result.

Let $X = (X_t)_{t\geq 0}$ be a minimal continuous-time Markov chain on a countable state space E with generator $\mathbf{G}$. We say that state $i \in E$ is *recurrent* if

$$\mathrm{P}(\{t \geq 0 : X_t = i\} \text{ is unbounded } | X_0 = i) = 1.$$

We say that state $i \in E$ is *transient* if

$$\mathrm{P}(\{t \geq 0 : X_t = i\} \text{ is unbounded } | X_0 = i) = 0.$$

We state the following results without proof:

- If state i is recurrent for the jump chain, then i is recurrent for X.
- If state i is transient for the jump chain, then i is transient for X.
- Every state is either recurrent or transient.
- Recurrence and transience are class properties.

See Norris (1998) p. 115 for a proof of the above result.

Let $X = (X_t)_{t\geq 0}$ be a minimal continuous-time Markov chain on a countable state space E with generator $\mathbf{G}$. Suppose $E = \{1, 2, 3, 4\}$ and

$$\mathbf{G} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{6} & 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For each state in the state space, decide whether it is recurrent or transient and justify your answer.

4.3 Prerequisites: Lecture 20

Exercise 4- 37: Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov chain on the state space $E = \{1, 2\}$ with generator

$$\mathbf{G} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

(a) Find the stationary distribution of X.

(b) Find the stationary distribution of the jump chain associated with X.

(c) Find the transition matrix $\mathbf{P}_t = (p_{ij}(t))_{i,j\in E}$ for all $t \geq 0$.
Hint: You may use without a proof that $\mathbf{G} = \mathbf{ODO}^{-1}$, where

$$\mathbf{O} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \mathbf{O}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

4.4 Prerequisites: Lecture 21

Exercise 4- 38: Let $\{N_t\}_{t\geq 0}$ be a birth process with intensities $\lambda_0, \lambda_1, \ldots$, such that $\lambda_i \neq \lambda_j$ for any $i \neq j$, and $N_0 = 0$. Derive the forward equations for this process. Hence verify that

$$p_n(t) = \frac{1}{\lambda_n} \sum_{i=0}^{n} \lambda_i e^{-\lambda_i t} \left[\prod_{j=0, j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i} \right]$$

where $p_n(t) = \mathrm{P}(N_t = n)$ and the convention $\prod_{\emptyset} = 1$ is used.

Exercise 4- 39: Consider a linear birth process $N = (N_t)_{t\geq 0}$ with birth rates given by $\lambda_n = n\lambda$, for $n \in \mathbb{N}$, $\lambda > 0$. Assume that $N_0 = 1$. Determine whether or not this birth process explodes and justify your answer.

4.5 Prerequisites: Lecture 22

Exercise 4- 40: Consider a population of N individuals consisting at time 0 of one 'infective' and $N-1$ 'susceptibles'. The process changes only by susceptibles becoming infective. We assume that this process can be modelled as a birth process. If, at some time t, there are i infectives, then, for each susceptible, there is a probability of $i\lambda\delta + o(\delta)$ of becoming infective in $(t, t+\delta]$ for $\lambda, \delta > 0$.

(a) If we consider the event of becoming an infective as a birth, what is the birth rate λ_i of the process, when there are i infectives?

(b) Let T denote the time to complete the epidemic, i.e. the first time when all N individuals are infective.

1. Derive $\mathrm{E}(T)$ (without using any type of generating functions).
2. Show that the Laplace transform of T is given by

$$\mathrm{E}[e^{-sT}] = \prod_{i=1}^{N-1} \left(\frac{\lambda_i}{\lambda_i + s} \right), \qquad \text{for } s \geq 0.$$

3. Derive $\mathrm{E}(T)$ by using the Laplace transform given in (2.).

You may leave your solution in (1.) and (3.) as a sum.

Exercise 4- 41: A colony of $N > 1$ creatures inhabit a planet which has continual daylight, and the pattern of waking and sleeping follows a continuous-time homogeneous Markov chain, more precisely, a birth-death process. The probability that a particular sleeping individual awakes during a time interval of length $[t, t+\delta]$ is $\beta\delta + o(\delta)$, and the probability that a particular awake individual falls asleep during a time interval $[t, t+\delta]$ of length is $\nu\delta + o(\delta)$. Assume that individuals behave independently of each other. We are interested in the number of individuals awake at time t.

(a) Find the generator matrix.

(b) Find the stationary distribution.

Consider the 2-state Markov chain (with states s sleep and w wake) for one individual with transition matrix

$$\mathbf{P}_t = \begin{pmatrix} 1 - p_{sw}(t) & p_{sw}(t) \\ 1 - p_{ww}(t) & p_{ww}(t) \end{pmatrix}.$$

(c) Write down the generator for this 2-state process.

(d) Calculate $p_{ww}(t)$ and $p_{sw}(t)$ using the forward equations.

(e) If $X_{m,t}$ denotes the number awake at time t given there are $m < N$ awake at time 0, what is $\mathrm{E}[X_{m,t}]$?

Exercise 4- 42: A population member alive at t dies during $(t, t+\delta)$ with probability $\mu\delta + o(\delta)$, independently of other population members. The population changes size only from the death of population members (there are no births, emigration or immigration). We assume that the population size can be modelled as a death process. The initial population size is n_0. Let T be the time at which the population dies out; i.e. $T = \min\{t \geq 0 : N_t = 0\}$. By considering the times between successive changes in population size find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$.

5 Problem sheet 5: Brownian motion

5.1 Prerequisites: Lecture 23

Exercise 5- 43: What is the autocovariance of a standard Brownian motion? I.e. compute $\mathrm{Cov}(W_t, W_s)$, for $s, t \geq 0$.

Exercise 5- 44: *Definition:* A stochastic process $(X_t)_{t\geq 0}$ is called a *Gaussian process* if $(X_{t_1}, \ldots, X_{t_n})$ has multivariate normal distribution for all $t_1, \ldots, t_n, n \in \mathbb{N}$.

Using the famous Cramer-Wold device, we have the following result: A random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top \in \mathbb{R}^n$ follows a multivariate normal distribution if $\mathbf{t}^\top \mathbf{X}$ follows a univariate normal distribution for all $\mathbf{t} = (t_1, \ldots, t_n)^\top \in \mathbb{R}^n$.

Show the following results:

1. A Brownian motion B is a Gaussian process with zero mean and $\mathrm{Cov}(B_t, B_s) = \min(t, s)$ for $s, t \geq 0$.
2. A Gaussian process $(X_t)_{t\geq 0}$ starting at zero, having continuous sample paths and having mean zero and covariance given by $\mathrm{Cov}(X_t, X_s) = \min(s, t)$ for all $s, t \geq 0$, is a Brownian motion.

Exercise 5- 45: Let W denote a standard Brownian motion. Show that

$$Y_t = W_1 - W_{1-t}, \quad 0 \leq t \leq 1,$$

is a standard Brownian motion for $0 \leq t \leq 1$. [Time reversal]

Exercise 5- 46: Let W denote a standard Brownian motion. Show that $(Y_t)_{t\geq 0}$ with $Y_0 = 0$ and

$$Y_t = tW_{1/t}, \quad t > 0,$$

is a standard Brownian motion. [Time inversion]
Hint: You may assume without proof that Y is continuous at 0.

Exercise 5- 47: Let $B = (B_t)_{t\geq 0}$ denote a standard Brownian motion. Let $a > 0$ denote a deterministic constant. Show that $W = (W_t)_{t\geq 0}$ with $W_t = aB_{t/a^2}$ is a standard Brownian motion.

Exercise 5- 48: Let $W = (W_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$ be independent standard Brownian motions. Let $\alpha, \beta \in \mathbb{R}$. Let

$$Y_t = \alpha W_t + \beta B_t,$$

for $t \geq 0$. Derive sufficient conditions on α, β to ensure Y is a standard Brownian motion.

References

Grimmett, G. R. & Stirzaker, D. R. (2001*a*), *One Thousand Exercises in Probability*, Oxford University Press, New York.

Grimmett, G. R. & Stirzaker, D. R. (2001*b*), *Probability and random processes*, third edn, Oxford University Press, New York.