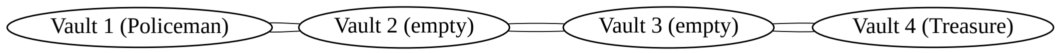
2 Problem sheet 2: Discrete-time Markov chains

2.1 Prerequisites: Lecture 7

Exercise 2- 14: Consider a sequence of 4 vaults labelled as $E = \{1, 2, 3, 4\}$. In vault 1 there is a policeman and in vault 4 there is a treasure chest. Vaults 2 and 3 are empty, see the picture below.



Suppose there is a thief walking through the vaults and you can model the location of the thief, i.e. the corresponding vault number, at each point in time by a homogeneous Markov chain denoted by $(X_n)_{n\in\{0,1,2,\dots\}}$.

If the thief is in vault 1 (together with the policeman), he will run out of vault 1 to vault 2 with probability one. If the thief is in vault 4 (with the treasure), he will stay there forever. If the thief is in vault 2 or 3, he will either go left with probability 1/2 or he will go right with probability 1/2. Moreover, assume that the thief is not very clever, so he might return to vault 1 (with the policeman) several times. Also, suppose the policeman never manages to catch the thief and jail him (even if they are in the same vault).

(a) State the transition probabilities for this Markov chain.

(b) Suppose the thief starts his journey in vault 1. What is the expected number of moves required until the thief reaches the treasure chest? Justify your answer carefully.

Solution:

1. The transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Let $T = \min\{n \geq 0 : X_n = 4\}$. Set $\nu_i = \mathrm{E}(T|X_0 = i)$ for $i \in E = \{1, 2, 3, 4\}$. We need to find ν_1.
We proceed recursively: Clearly, $\nu_4 = 0$. Similarly to the Gambler's ruin problem, we condition on the outcome of the first move. Also, we apply the law of the total conditional expectation, the Markov property and time-homogeneity:

$$\begin{aligned}
\nu_3 &= \mathrm{E}(T|X_0 = 3) \\
&= \sum_{x_1=1}^{4} \mathrm{E}(T|X_0 = 3, X_1 = x_1)\mathrm{P}(X_1 = x_1|X_0 = 3) \\
&= \mathrm{E}(T|X_0 = 3, X_1 = 2)\mathrm{P}(X_1 = 2|X_0 = 3) + \mathrm{E}(T|X_0 = 3, X_1 = 4)\mathrm{P}(X_1 = 4|X_0 = 3) \\
&= \frac{1}{2}\mathrm{E}(T|X_1 = 2) + \frac{1}{2}\mathrm{E}(T|X_1 = 4) = \frac{1}{2}[\mathrm{E}(T|X_0 = 2) + 1] + \frac{1}{2}[\mathrm{E}(T|X_0 = 4) + 1] \\
&= 1 + \frac{1}{2}(\nu_2 + \nu_4) = 1 + \frac{1}{2}\nu_2, \\
\nu_2 &= \sum_{x_1=1}^{4} \mathrm{E}(T|X_0 = 2, X_1 = x_1)\mathrm{P}(X_1 = x_1|X_0 = 2) \\
&= \mathrm{E}(T|X_0 = 2, X_1 = 1)\mathrm{P}(X_1 = 1|X_0 = 2) + \mathrm{E}(T|X_0 = 2, X_1 = 3)\mathrm{P}(X_1 = 3|X_0 = 2) \\
&= \frac{1}{2}(1 + \nu_1) + \frac{1}{2}(1 + \nu_3) = 1 + \frac{1}{2}(\nu_1 + \nu_3),
\end{aligned}$$

$$\nu_1 = \sum_{x_1=1}^{4} \mathrm{E}(T|X_0=1, X_1=x_1)\mathrm{P}(X_1=x_1|X_0=1)$$
$$= \mathrm{E}(T|X_0=1, X_1=2)\mathrm{P}(X_1=2|X_0=1) = 1+\nu_2.$$

We have $\nu_1 = 1+\nu_2, \nu_3 = 1+\frac{1}{2}\nu_2$ and

$$\nu_2 = 1+\frac{1}{2}(\nu_1+\nu_3) = 1+\frac{1}{2}\left(1+\nu_2+1+\frac{1}{2}\nu_2\right) = 2+\frac{3}{4}\nu_2 \Leftrightarrow \nu_2 = 8.$$

Hence, $\nu_1 = 9$ (and also $\nu_3 = 5$). I.e. the expected number of moves required until the thief reaches the treasure chest is 9.

Let us check some of the details required in the above computations: E.g. in the calculations above, we claimed that

$$\mathrm{E}(T|X_1=2) = \mathrm{E}(T|X_0=2)+1$$

and similar results appeared subsequently. To see that the stated result hold, we can spell out the details of the computation as follows:

$$\mathrm{E}(T|X_1=2) = \sum_{t=0}^{\infty} t\mathrm{P}(T=t|X_1=2) = \sum_{t=1}^{\infty} t\mathrm{P}(T=t|X_1=2) \text{ (next replace t by t + 1)}$$
$$= \sum_{t+1=1}^{\infty}(t+1)\mathrm{P}(T=t+1|X_1=2) = \sum_{t=0}^{\infty}(t+1)\mathrm{P}(T=t+1|X_1=2)$$
$$= \sum_{t=0}^{\infty} t\mathrm{P}(T=t+1|X_1=2) + \sum_{t=0}^{\infty}\mathrm{P}(T=t+1|X_1=2).$$

We note that, for the first term, we get

$$\sum_{t=0}^{\infty} t\mathrm{P}(T=t+1|X_1=2) = \sum_{t=1}^{\infty} t\mathrm{P}(T=t+1|X_1=2)$$
$$= \sum_{t=1}^{\infty} t\mathrm{P}(T=t|X_0=2) = \sum_{t=0}^{\infty} t\mathrm{P}(T=t|X_0=2) = \mathrm{E}(T|X_0=2),$$

where we used the time-homogeneity of the Markov chain in the second equality.

For the second term, we have

$$\sum_{t=0}^{\infty}\mathrm{P}(T=t+1|X_1=2) = \sum_{t=1}^{\infty}\mathrm{P}(T=t|X_1=2) = \sum_{t=0}^{\infty}\mathrm{P}(T=t|X_1=2) = 1,$$

where we replaced t by $t-1$ in the first equality and used the fact that

$$\mathrm{P}(T=0|X_1=2) = \mathrm{P}(X_0=4|X_1=2) \overset{\text{Bayes}}{=} \frac{\mathrm{P}(X_1=2|X_0=4)\mathrm{P}(X_0=4)}{\mathrm{P}(X_1=2)} = 0,$$

since $p_{42} = 0$.

2.2 Prerequisites: Lecture 9

Exercise 2- 15: A transition matrix is called *doubly stochastic* if all its column sums equal 1, that is, if $\sum_i p_{ij} = 1$ for all $i, j \in E$.

(a) Assume the Markov chain has finite state space, i.e. $|E| = K < \infty$.

- Show that if the transition matrix is doubly stochastic, then all states are positive recurrent.
- Show that if the transition matrix is doubly stochastic and, in addition, if the chain is irreducible and aperiodic, then $p_{ij}(n) \to \frac{1}{K}$ as $n \to \infty$.

(b) Assume the Markov chain has infinite state space, i.e. $|E| = \infty$.

- Show that if the chain is irreducible and the transition matrix is doubly stochastic, then all states are either null recurrent or transient.

Solution:

(a) Let $\mathbf{P}$ be the doubly stochastic transition matrix.

- Then

$$\sum_{i\in E} p_{ij}(n) = \sum_{i\in E}\sum_{k\in E} p_{ik}(n-1)p_{kj} = \sum_{k\in E}\left(\sum_{i\in E} p_{ik}(n-1)\right)p_{kj}$$

where we used the CK equations. Now we can prove by induction that $\mathbf{P}^n$ is also doubly stochastic for all $n \in \mathbb{N}$.

Suppose $j \in E$ is *not* positive recurrent. Then $p_{ij}(n) \to 0$ as $n \to \infty$ for all $i \in E$. Then $1 = \sum_i p_{ij}(n) \to 0$ (we can interchange limit and sum since we have a finite chain). This is a contradiction! Hence all states are positive recurrent.

- In addition assume the chain is irreducible and aperiodic, then $p_{ij}(n) \to \pi_j$, where $\boldsymbol{\pi}$ is the unique stationary distribution. Since $\mathbf{P}$ is doubly stochastic we get for $\boldsymbol{\pi} := (1/K, \dots, 1/K)$, that $\pi_i \geq 0$, $\sum_{i\in E} \pi_i = 1$ and

$$\sum_{i\in E} \pi_i p_{ij} = \frac{1}{K}\sum_{i\in E} p_{ij} = \frac{1}{K} = \pi_j.$$

(b) We only need to show that the chain cannot be positive recurrent.

Suppose the chain is positive recurrent. Then according to Theorem 3.9.8 there exists a positive root of the equation $\mathbf{xP} = \mathbf{x}$, which is unique up to a multiplicative constant. Since $\mathbf{P}$ is doubly stochastic, we can take $\mathbf{x} = \mathbf{1}$ (the vector of 1's). Since the root $\mathbf{x}$ is unique, there cannot exist a stationary *distribution* and therefore the chain is null or transient.

Exercise 2- 16: The following question is adapted from Grimmett & Stirzaker (2001*b*,*a*) Problem 6.15.7:

Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a recurrent irreducible Markov chain on the state space E with transition matrix $\mathbf{P}$, and let $\mathbf{x}$ be a positive solution of the equation $\mathbf{x} = \mathbf{xP}$.

(a) Show that

$$q_{ij}(n) = \frac{x_j}{x_i} p_{ji}(n), \qquad i, j \in E, n \in \mathbb{N},$$

defines the n-step transition probabilities of a recurrent irreducible Markov chain on E whose first-passage probabilities are given by

$$g_{ij}(n) = \frac{x_j}{x_i} l_{ji}(n), \qquad i \neq j, n \in \mathbb{N}, \tag{2.1}$$

where $l_{ji}(n) = \mathrm{P}(X_n = i, T_j \geq n | X_0 = j)$ and $T_j = \min\{m \in \mathbb{N} : X_m = j\}$.

(b) Show that $\mathbf{x}$ is unique up to a multiplicative constant.

Solution:

(a) We observe that $q_{ij}(n) \geq 0$ for all $i, j \in E, n \in \mathbb{N}_0$. Also,

$$\sum_{j\in E} q_{ij}(n) = \sum_{j\in E} \frac{x_j}{x_i} p_{ji}(n) = \frac{1}{x_i} \sum_{j\in E} x_j p_{ji}(n) = \frac{x_i}{x_i} = 1,$$

for all $i \in E, n \in \mathbb{N}_0$. Hence $\mathbf{Q}(n)$ is indeed a stochastic matrix. Also,

$$q_{ij}(n+1) = \frac{x_j}{x_i} p_{ji}(n+1) \overset{\text{Chapman-Kolmogorov}}{=} \frac{x_j}{x_i} \sum_{l\in E} p_{jl}(n) p_{li}$$
$$= \sum_{l\in E} \frac{x_j}{x_l} p_{jl}(n) \frac{x_l}{x_i} p_{li} = \sum_{l\in E} q_{lj}(n) q_{il},$$

for all $i, j \in E, n \in \mathbb{N}_0$. Hence $\mathbf{Q}(1)$ is the transition matrix of a Markov chain $\{Y_n\}_{n\in\mathbb{N}_0}$, say, and $\mathbf{Q}(n) = \mathbf{Q}^n$. The chain $\{Y_n\}_{n\in\mathbb{N}_0}$ is also recurrent since

$$\sum_{n=1}^{\infty} q_{ii}(n) = \sum_{n=1}^{\infty} \frac{x_i}{x_i} p_{ii}(n) = \sum_{n=1}^{\infty} p_{ii}(n) = \infty,$$

for all $i \in E$. Also, $\{Y_n\}_{n\in\mathbb{N}_0}$ is irreducible since $i \to j$ for $\{Y_n\}_{n\in\mathbb{N}_0}$ whenever $j \to i$ for $\{X_n\}_{n\in\mathbb{N}_0}$, and $\{X_n\}_{n\in\mathbb{N}_0}$ is irreducible.

Next we compute the first passage probabilities of $\{Y_n\}_{n\in\mathbb{N}_0}$ which we denote by $g_{ij}(n)$ for $i \neq j$. We conduct a proof by induction. The claim is true for $n = 1$, since we have

$$g_{ij}(1) = q_{ij}(1) = \frac{x_j}{x_i} p_{ji}(1) = \frac{x_j}{x_i} l_{ji}(1).$$

Now suppose the claim is true for $n \in \mathbb{N}$, then, by equation (3.9.5) in the lecture notes

$$l_{ji}(n+1) = \sum_{r\in E: r\neq j} p_{ri} l_{jr}(n).$$

Hence,

$$\frac{x_j}{x_i} l_{ji}(n+1) = \sum_{r\in E: r\neq j} \frac{x_r}{x_i} p_{ri} \frac{x_j}{x_r} l_{jr}(n) = \sum_{r\in E: r\neq j} q_{ir} g_{rj}(n) = g_{ij}(n+1),$$

where we applied the law of total probability, the Markov property and time-homogeneity in the last step. More precisely, we used that

$$f_{ij}(n+1) = \sum_{r: r\neq j} p_{ir} f_{rj}(n), \quad \text{for } i \neq j, n \in \mathbb{N}.$$

To see that, note that for $i \neq j, n \in \mathbb{N}$

$$f_{ij}(n+1) = \mathrm{P}(T_j = n+1 | X_0 = i) \overset{LTP}{=} \sum_{r: r\neq j} \mathrm{P}(T_j = n+1 | X_1 = r, X_0 = i) \mathrm{P}(X_1 = r | X_0 = i)$$
$$\overset{\text{Markov}}{=} \sum_{r: r\neq j} \mathrm{P}(T_j = n+1 | X_1 = r) \mathrm{P}(X_1 = r | X_0 = i)$$
$$\overset{\text{time-homogeneity}}{=} \sum_{r: r\neq j} \mathrm{P}(T_j = n | X_0 = r) \mathrm{P}(X_1 = r | X_0 = i)$$
$$= \sum_{r: r\neq j} f_{rj}(n) p_{ir}.$$

(b) We sum (2.1) over n and obtain for the left hand side (LHS):

$$\text{LHS} = \sum_{n=1}^{\infty} g_{ij}(n) = g_{ij} = 1,$$

by Exercise 1- 13. And for the right hand side (RHS), we obtain

$$\text{RHS} = \sum_{n=1}^{\infty} \frac{x_j}{x_i} l_{ji}(n) = \frac{x_j}{x_i} \sum_{n=1}^{\infty} l_{ji}(n) = \frac{x_j}{x_i} \rho_i(j).$$

Hence

$$\text{LHS} = \text{RHS} \Leftrightarrow x_i = x_j \rho_i(j), \qquad \forall i \neq j.$$

Without loss of generality assume that $0 \in E$, then $x_i = x_0 \rho_i(0)$ for all $i \in E$. Hence $\mathbf{x}$ is unique up to a multiplicative constant.

Exercise 2- 17: Let T be a nonnegative integer-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and let $A \in \mathcal{F}$ be an event with $\mathrm{P}(A) > 0$. Show that

$$\mathrm{E}(T|A) = \sum_{n=1}^{\infty} \mathrm{P}(T \geq n|A).$$

Solution: We use the definition of the conditional expectation and the fact that $m = \sum_{n=0}^{m-1} 1$ to deduce that

$$\begin{aligned}\mathrm{E}(T|A) &= \sum_{m=0}^{\infty} m\mathrm{P}(T = m|A) = \sum_{m=0}^{\infty} \sum_{n=0}^{m-1} \mathrm{P}(T = m|A) = \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \mathrm{P}(T = m|A) \\ &= \sum_{n=0}^{\infty} \mathrm{P}(T \geq n+1|A) = \sum_{n=1}^{\infty} \mathrm{P}(T \geq n|A).\end{aligned}$$

2.3 Prerequisites: Lecture 10

Exercise 2- 18: We consider Markov chains with state space $E = \{0, 1, 2, 3\}$. For each of the Markov transition matrices below:

$$(a) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (b) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix} \quad (c) \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

- Specify the communicating classes and determine whether they are transient or recurrent;
- Decide whether or not they have a unique stationary distributions;
- Find a stationary distribution for each of them and show that it is not unique where appropriate.

Solution:

(a)
- We have three transient classes $\{0\}, \{1\}, \{2\}$ and 1 closed recurrent class $\{3\}$, (the state 3 is absorbing).
- Since we have only 1 closed communicating class: $C := \{3\}$ on a finite state space, (the state 3 is absorbing), there is a unique stationary distribution!
- Compute the unique stationary solution $\pi_C = \pi_C P_C$. Here: $P_C = 1$, and $\pi_C = 1$. Now we set $\boldsymbol{\pi} = (0, 0, 0, 1)$.

(b)
- We have 2 closed recurrent classes $C_1 = \{0, 1\}$ and $C_2 = \{2, 3\}$.
- Since we have 2 closed communicating classes $C_1 = \{0, 1\}$ and $C_2 = \{2, 3\}$ on a finite state space, there is a stationary solution, but it is not unique!
- We get $\boldsymbol{\pi} = (a, a, b, b)$ for $a, b \geq 0$ with $2(a + b) = 1$.

(c)
- There is 1 closed, recurrent class: $C = \{0, 1, 2\}$ and one transient class $\{3\}$.
- Since there is only 1 closed communicating class: $C = \{0, 1, 2\}$ on a finite state space, there is a unique stationary solution.
- Solve $\boldsymbol{\pi}_C = \boldsymbol{\pi}_C \mathbf{P}_C$! Then we obtain $\boldsymbol{\pi}_C = (1/4, 5/12, 1/3)$ and $\boldsymbol{\pi} = (1/4, 5/12, 1/3, 0)$.

Exercise 2- 19: Consider a discrete-time homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0.75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Draw the transition diagram.

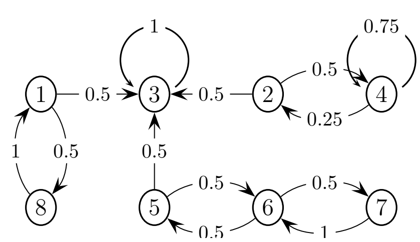
(b) Specify the communicating classes and determine whether they are transient, null recurrent or positive recurrent. *Please note that you need to justify your answers.*

(c) Find all stationary distributions.

(d) For each communication class, pick a state i and find the first passage probabilities $f_{ii}(n) = \mathrm{P}(X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i | X_0 = i)$ for all $n \in \mathbb{N}$ and derive $f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n)$.

Solution:

(a) The transition diagram is given by



(b) We have a finite state space with four communicating classes: The classes $T_1 = \{1,8\}, T_2 = \{2,4\}, T_3 = \{5,6,7\}$ are not closed and hence transient. The class $C_1 = \{3\}$ is finite and closed and hence positive recurrent.

(c) According to a theorem from lectures, this Markov chain has a unique stationary distribution $\boldsymbol{\pi}$ since it has one closed communicating classes in a finite state space. For the transient states we know from the lectures that $\pi_i = 0$ for $i = 1,2,4,5,6,7,8$. Hence $\boldsymbol{\pi} = (0,0,1,0,0,0,0,0)$ is the unique stationary distribution.

(d) For each communicating class, we pick one state, e.g.

T_1: $f_{11}(1) = 0, f_{11}(2) = 0.5, f_{11}(n) = 0, \forall n \geq 3 \Rightarrow f_{11} = 0.5.$
$f_{88}(1) = 0, f_{88}(2) = 0.5, f_{88}(n) = 0, \forall n \geq 3 \Rightarrow f_{88} = 0.5.$

T_2: $f_{22}(1) = 0, f_{22}(2) = 0.5 \times 0.25 = 1/8, f_{22}(3) = 0.5 \times 0.75 \times 0.25 = 3/32, f_{22}(n) = \frac{1}{2}\left(\frac{3}{4}\right)^{n-2}\frac{1}{4}, \forall n \geq 2 \Rightarrow f_{22} = 0.5.$
$f_{44}(1) = 0.75, f_{44}(2) = 0.5 \times 0.25 = 1/8, f_{44}(n) = 0, \forall n \geq 3 \Rightarrow f_{44} = 7/8.$

T_3: $f_{55}(1) = 0, f_{55}(2) = 0.5 \times 0.5 = 0.25, f_{55}(3) = 0, f_{55}(4) = 0.5^3 = 1/8, \ldots,$, i.e. $f_{55}(n) = 0$ for odd n and $f_{55}(n) = 0.5^{\frac{n}{2}+1}$ for even n. Hence $f_{55} = 0.5$.
$f_{66}(1) = 0, f_{66}(2) = 0.5 \times 0.5 + 0.5 \times 1 = 3/8, f_{66}(n) = 0, \forall n \geq 3 \Rightarrow f_{66} = 3/8.$
$f_{77}(1) = 0, f_{77}(2) = 0.5, f_{77}(3) = 0, f_{77}(4) = 0.5^3 = 1/8, \ldots$, i.e. $f_{77}(n) = 0$ for odd n and $f_{77}(n) = 0.5^{n-1}$ for even n. Hence $f_{77} = 2/3$.

C_1: $f_{33}(1) = 1, f_{33}(n) = 0, \forall n \geq 2 \Rightarrow f_{33} = 1.$

Exercise 2- 20: Suppose we have a Markov chain with finite state space E, i.e. $K = |E| < \infty$, and transition matrix $\mathbf{P}$. Suppose for some $i \in E$ that

$$p_{ij}(n) \to \pi_j \text{ as } n \to \infty \text{ for all } j \in E.$$

Then $\boldsymbol{\pi}$ is a stationary distribution.

Solution: We have $\pi_j \geq 0$ and

$$\sum_{j \in E} \pi_j = \sum_{j \in E} \lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} \sum_{j \in E} p_{ij}(n) = 1,$$

since $\mathbf{P}_n$ is stochastic. Also

$$\pi_j = \lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} \sum_{k \in E} p_{ik}(n-1)p_{kj} = \sum_{k \in E} \lim_{n\to\infty} p_{ik}(n-1)p_{kj}$$
$$= \sum_{k \in E} \pi_k p_{kj},$$

where we used the CK equations. Note that we have used the finiteness of E to justify the interchange of summation and limit operations.

Exercise 2- 21: Consider a discrete-time homogeneous Markov chain $(X_n)_{n \in \mathbb{N}_0}$ with state space $E = \{1,2\}$ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

(a) Derive $\lim_{n\to\infty} \mathrm{P}(X_n = i)$ for $i \in \{1,2\}$.

(b) Find

$$\lim_{N\to\infty} \frac{1}{N+1} \mathrm{E}\left(\sum_{n=0}^{N} e^{X_n}\right).$$

Hint: You may use the following result from Analysis without a proof: Let $(x_n)_{n\in\mathbb{N}_0}$ be a real-valued convergent sequence with $\lim_{n\to\infty} x_n = x$. Then $\lim_{m\to\infty} \frac{1}{m+1}\sum_{n=0}^{m} x_n = x$.

Solution:

(a) We can read off from the transition matrix that this Markov chain is irreducible and aperiodic with a finite state space. Hence we conclude from the lectures that there exists a unique stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2)$ and that the limiting distribution is given by the stationary distribution.

We derive the stationary distribution: $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}, \pi_1 + \pi_2 = 1, \Leftrightarrow \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 = \pi_1, \pi_1 + \pi_2 = 1, \Leftrightarrow \frac{1}{4}\pi_2 = \frac{1}{2}\pi_1, \pi_1 + \pi_2 = 1 \Leftrightarrow \pi_2 = 2\pi_1, \pi_2 = 1 - \pi_1 \Leftrightarrow \pi_1 = \frac{1}{3}, \pi_2 = \frac{2}{3}$. Hence, $\lim_{n\to\infty} \mathrm{P}(X_n = 1) = \frac{1}{3}$ and $\lim_{n\to\infty} \mathrm{P}(X_n = 2) = \frac{2}{3}$.

(b) Using the linearity of the expectation, we have

$$\mathrm{E}\left(\sum_{n=0}^{N} e^{X_n}\right) = \sum_{n=0}^{N} \mathrm{E}\left(e^{X_n}\right) = \sum_{n=0}^{N}\sum_{k=1}^{2} e^k \mathrm{P}(X_n = k) = \sum_{k=1}^{2} e^k \sum_{n=0}^{N} \mathrm{P}(X_n = k).$$

Hence we have

$$\lim_{N\to\infty} \frac{1}{N+1} \mathrm{E}\left(\sum_{n=0}^{N} e^{X_n}\right) = \sum_{k=1}^{2} e^k \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \mathrm{P}(X_n = k)$$

$$\overset{Hint}{=} \sum_{k=1}^{2} e^k \lim_{n\to\infty} \mathrm{P}(X_n = k) = e\pi_1 + e^2\pi_2 = \frac{1}{3}e + \frac{2}{3}e^2.$$