3 Problem sheet 3: Exponential distribution and Poisson processes

3.1 Prerequisites: Lecture 12

Exercise 3- 22: Prove Theorem 4.1.4: Let $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n > 0$. Consider independent random variables $H_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \dots, n$. Let $H := \min\{H_1, \dots, H_n\}$. Then

(a) $H \sim \text{Exp}(\sum_{i=1}^n \lambda_i)$.

(b) For any $k = 1, \dots, n$, $\text{P}(H = H_k) = \lambda_k / (\sum_{i=1}^n \lambda_i)$.

Solution:

(a) Let $y > 0$. Then

$$\begin{aligned}\text{P}(H > y) = \text{P}(H_1 > y, \dots, H_n > y) &\overset{\text{independence}}{=} \prod_{i=1}^n \text{P}(H_i > y)\\ &= \prod_{i=1}^n e^{-\lambda_i y} = \exp\left(-y \sum_{i=1}^n \lambda_i\right).\end{aligned}$$

For $y \le 0$, we have $\text{P}(H > y) = 1$. Hence the functional form of the survival function of H is the one of an exponentially distributed random variable with parameter $\sum_{i=1}^n \lambda_i$.

(b) For any $k = 1, \dots, n$, we have

$$\begin{aligned}\text{P}(H = H_k) &= \text{P}(H_1 \ge H_k, \dots, H_n \ge H_k)\\ &\overset{\text{LTP}}{=} \int_0^\infty \text{P}(H_1 \ge H_k, \dots, H_n \ge H_k | H_k = x) f_{H_k}(x) dx\\ &= \int_0^\infty \text{P}(H_1 \ge x, \dots, H_n \ge x | H_k = x) f_{H_k}(x) dx\\ &\overset{\text{independence}}{=} \int_0^\infty \text{P}(H_1 \ge x) \cdots \text{P}(H_n \ge x) f_{H_k}(x) dx\\ &= \int_0^\infty \exp\left(-x \sum_{i=1}^n \lambda_i\right) \lambda_k dx = \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}.\end{aligned}$$

Exercise 3- 23: A lump of radioactive material has 10^{26} atoms. Each atom decays after an exponential time with mean 10^{26} seconds, independently of the decay of other atoms.

(a) Let X_1 be the time taken for the first decay to occur. Why is $X_1 \sim \text{Exp}(1)$? Let X_2 be the time between the first decay and the second. What is the distribution of X_2?

(b) What is the probability a given atom decays in the first minute? Let N be the total number of decays in the first minute. By noting that N has a binomial distribution, show that N has an approximate Poisson distribution with mean 60.

Solution:

(a) Let W_i be the time taken for atom i to decay. Then $X_1 = \min\{W_1, W_2, \dots, W_{10^{26}}\}$. Since $W_i \sim \text{Exp}(10^{-26})$ and the W_i's are independent, $X_1 \sim \text{Exp}(1)$. To see that note that, for $x > 0$,

$$\text{P}(X_1 > x) = \text{P}(\min\{W_1, W_2, \dots, W_{10^{26}}\} > x)$$

$$= \mathrm{P}(W_1 > x, W_2 > x \ldots, W_{10^{26}} > x) = \prod_{i=1}^{10^{26}} \mathrm{P}(W_i > x)$$
$$= (\mathrm{P}(W_1 > x))^{10^{26}} = \left(e^{-10^{-26}x}\right)^{10^{26}} = e^{-x}.$$

Also, $\mathrm{P}(X_1 > x) = 1$ for $x \le 0$. Hence $X_1 \sim \mathrm{Exp}(1)$.

After the first atom decays there are $10^{26} - 1$ left. Without loss of generality we assume that the atom with index $i = 10^{26}$ decayed first (otherwise we could just re-shuffle the indices). By the lack of memory property each remaining atom decays after a further time which is $\mathrm{Exp}(10^{-26})$. For $x > 0$,

$$\mathrm{P}(X_2 > x) = \mathrm{P}(\min\{W_1, W_2, \ldots, W_{10^{26}-1}\} > x) = \left(e^{-10^{-26}x}\right)^{10^{26}-1}$$
$$= \left(e^{-10^{-26}(10^{26}-1)x}\right) = \left(e^{-(1-10^{-26})x}\right),$$

and $\mathrm{P}(X_2 > x) = 1$ for $x \le 0$. Hence $X_2 \sim \mathrm{Exp}(1 - 10^{-26})$. [Note that we could have used Theorem 4.1.4 instead of deriving the above result.]

(b) Let $p = \mathrm{P}(W_i < 60) = 1 - \exp\{-60 \times 10^{-26}\}$, so that $N \sim \mathrm{Binomial}(10^{26}, p)$. Recall that the Poisson distribution with parameter $\lambda = np$ approximates the binomial distribution $\mathrm{Binomial}(n, p)$ if n is sufficiently large and p is sufficiently small. In our cases, since 10^{26} is enormous and p very small, the distribution of N is approximately Poisson with mean $\lambda = 10^{26}p$. But

$$p = 1 - (1 - 60 \times 10^{-26} + \frac{1}{2}(60 \times 10^{-26})^2 + \cdots) \approx 60 \times 10^{-26}.$$

Thus $\lambda \approx 60$.

3.2 Prerequisites: Lecture 15

Exercise 3- 24: Let $X_1, \ldots, X_n$ be i.i.d. random variables, following the uniform distribution on $(0, t)$ for $t > 0$. Let $X_{(1)}, \ldots, X_{(n)}$ denote the corresponding order statistics. That is, they have joint density function

$$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)}, \ldots, x_{(n)}) = \begin{cases} \frac{n!}{t^n}, & \text{if } 0 < x_{(1)} < \cdots < x_{(n)} < t, \\ 0, & \text{otherwise.} \end{cases}$$

Let $1 \le k \le n$; show that

$$\mathrm{E}[X_{(k)}] = \frac{tk}{n+1}.$$

Note that you do not have to derive the marginal density of $X_{(k)}$ from first principles.

Solution: *A small background on order statistics:*

Let us assume we have n i.i.d. random variables $X_1, \ldots, X_n$ with common density function f. For each $\omega \in \Omega$ order the realisation $X_1(\omega), \ldots, X_n(\omega)$ in non–decreasing order to obtain $X_{(1)}(\omega), \ldots, X_{(n)}(\omega)$. We call the new random variables $X_{(1)}, \ldots, X_{(n)}$ the *order statistics*.

Now we define for $r = 1, \ldots, n$

$$\mathbb{I}_r(\omega) := \mathbb{I}_{\{X_r(\omega) \le x\}}(\omega).$$

Note that $\mathbb{I}_r$ is a Bernoulli random variable with success probability

$$\mathrm{P}(\{X_r(\omega) \leq x\}) = F_X(x).$$

Then

$$S := \mathbb{I}_1 + \cdots + \mathbb{I}_n,$$

is a sum of independent Bernoulli random variables, which has Binomial ($Bin(n, F_X(x))$) distribution.

Then for $k = 1, \ldots, n$:

$$\mathrm{P}(X_{(k)} \leq x) = \mathrm{P}(S \geq k),$$

since the event $\{X_{(k)} \leq x\}$ indicates that at least k elements of the sample are smaller or equal to x. Then

$$\mathrm{P}(X_{(k)} \leq x) = \mathrm{P}(S \geq k) = \sum_{l=k}^{n} \binom{n}{l} (F_X(x))^l (1 - F_X(x))^{n-l}. \tag{3.1}$$

Then we differentiate (3.1), and we obtain the probability density function of the kth order statistic:

$$\begin{aligned}
f_{X_{(k)}}(x) &= \sum_{l=k}^{n} \binom{n}{l} \left[l F_X(x))^{l-1} f_X(x)(1 - F_X(x))^{n-l} \right. \\
&\qquad \left. + (n-l)(F_X(x))^l (1 - F_X(x))^{n-l-1}(-f_X(x)) \right] \\
&= \binom{n}{k} k F_X(x))^{k-1} (1 - F_X(x))^{n-k} f_X(x) + \sum_{l=k+1}^{n} \binom{n}{l} l (F_X(x))^{l-1} (1 - F_X(x))^{n-l} f_X(x) \\
&\quad - \sum_{l=k}^{n-1} \binom{n}{l} (n-l)(F_X(x))^l (1 - F_X(x))^{n-l-1} f_X(x) (\text{since the term l = n in the last sum} = 0) \\
&= \frac{n!}{(k-1)!(n-k)!} f_X(x)(F_X(x))^{k-1}(1 - F_X(x))^{n-k} \\
&\quad + \sum_{l=k}^{n-1} \binom{n}{l+1} (l+1)(F_X(x))^l (1 - F_X(x))^{n-l-1} f_X(x) \ (\text{replaced l by l} + 1) \\
&\quad - \sum_{l=k}^{n-1} \binom{n}{l} (n-l)(F_X(x))^l (1 - F_X(x))^{n-l-1} f_X(x) \\
&= \frac{n!}{(k-1)!(n-k)!} (F_X(x))^{k-1}(1 - F_X(x))^{n-k} f_X(x),
\end{aligned}$$

since

$$\binom{n}{l+1}(l+1) = \frac{n!}{l!(n-l-1)!} = \binom{n}{l}(n-l),$$

the last two sums just cancel out.

Note that in the case that the $X_1, \ldots, X_n$ are uniformly distributed on the interval $[0, t]$, then we have for $0 \leq x \leq t$:

$$F_X(x) = \frac{x}{t}, \qquad f_X(x) = \frac{1}{t}.$$

For this particular problem, it would be sufficient to start here (you don't need to derive the probability density of the order statistic).

We note that

$$f(x_{(k)}) = \frac{n!}{(k-1)!(n-k)!}\left(\frac{x_{(k)}}{t}\right)^{k-1}\left(1-\frac{x_{(k)}}{t}\right)^{n-k}\frac{1}{t}$$

which is a standard result for order statistics. Then the solution is a simple exercise in integration:

$$\begin{aligned}
\mathrm{E}[X_{(k)}] &= \frac{n!}{(k-1)!(n-k)!}\int_0^t x_{(k)}\left(\frac{x_{(k)}}{t}\right)^{k-1}\left(1-\frac{x_{(k)}}{t}\right)^{n-k}\frac{1}{t}dx_{(k)} \\
&= \frac{n!}{(k-1)!(n-k)!}t\int_0^1 u^k(1-u)^{n-k}du \\
&= B(k, n-k+1)^{-1}t\int_0^1 uu^{k-1}(1-u)^{n-k}du,
\end{aligned}$$

where

$$B(k, n-k+1) = \frac{\Gamma(k)\Gamma(n-k+1)}{\Gamma(n+1)},$$

and Γ is the Gamma function.

The latter integral is the expectation of a Beta random variable of parameters k and $n-k+1$ which is equal to $k/(n+1)$, so it follows

$$\begin{aligned}
\mathrm{E}[X_{(k)}] &= \frac{n!}{(k-1)!(n-k)!}t\frac{(k-1)!(n-k)!}{n!}\frac{k}{n+1} \\
&= \frac{tk}{n+1}.
\end{aligned}$$

Exercise 3- 25: Show that Definition 5.4.4 of a Poisson process implies Definition 5.3.4.

Solution: We check the four conditions of Definition 5.3.4:

1. $N_0 = \sup\{n \in \mathbb{N}_0 : J_n \leq 0\} = 0$.
2. We show that $N_t \sim \mathrm{Poi}(\lambda t)$ for all $t \geq 0$: Let $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
\mathrm{P}(N_t = n) &= \mathrm{P}(J_n \leq t < J_n + H_{n+1}) \\
&= \mathrm{P}(J_n \leq t, H_{n+1} > t - J_n) \\
&\overset{\text{LTP}}{=} \int_0^\infty \mathrm{P}(J_n \leq t, H_{n+1} > t - J_n | J_n = s) f_{J_n}(s) ds \\
&= \int_0^\infty \mathrm{P}(s \leq t, H_{n+1} > t - s | J_n = s) f_{J_n}(s) ds \\
&\overset{H_{n+1}, J_n \text{ indep.}}{=} \int_0^t \mathrm{P}(H_{n+1} > t - s) f_{J_n}(s) ds \\
&= \int_0^t e^{-\lambda(t-s)}\frac{\lambda^n}{\Gamma(n)}s^{n-1}e^{-\lambda s}ds \\
&= e^{-\lambda t}\int_0^t \frac{\lambda^n}{(n-1)!}s^{n-1}ds = \frac{(\lambda t)^n}{n!}e^{-\lambda t}.
\end{aligned}$$

3. We show the independence and stationarity of the increments. We first focus on only two increments: Let $0 \le s$ and let $t > 0$. We want to show that

$$\mathrm{P}(N_s = i, N_{s+t} - N_s = j) = \mathrm{P}(N_s = i)\mathrm{P}(N_t = j),$$

for all $i, j \in \mathbb{N}_0$.

- Case: $i = 0, j = 0$.

$$\begin{aligned}\mathrm{P}(N_s = 0, N_{s+t} - N_s = 0) &= \mathrm{P}(N_{s+t} = 0) = \mathrm{P}(J_1 > s+t) = \mathrm{P}(H_1 > s+t)\\ &= e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t} = \mathrm{P}(N_s = 0)\mathrm{P}(N_t = 0).\end{aligned}$$

- Case: $i = 0, j = 1$.

$$\begin{aligned}&\mathrm{P}(N_s = 0, N_{s+t} - N_s = 1) = \mathrm{P}(N_s = 0, N_{s+t} = 1)\\ &= \mathrm{P}(s < J_1 \le s+t < J_2) = \mathrm{P}(s < H_1 \le s+t, H_1 + H_2 > s+t)\\ &= \mathrm{P}(s < H_1 \le s+t, H_2 > s+t-H_1)\\ &\overset{\text{LTP}}{=} \int_0^\infty \mathrm{P}(s < H_1 \le s+t, H_2 > s+t-H_1 | H_1 = x) f_{H_1}(x)dx\\ &\overset{H_1,H_2\text{ indep.}}{=} \int_s^{s+t} \mathrm{P}(H_2 > s+t-x) f_{H_1}(x)dx\\ &= \int_s^{s+t} e^{-\lambda(s+t-x)}\lambda e^{-\lambda x}dx = \int_s^{s+t} e^{-\lambda(s+t)}\lambda dx\\ &= e^{-\lambda(s+t)}\lambda t = e^{-\lambda s}e^{-\lambda t}\lambda t = \mathrm{P}(N_s = 0)\mathrm{P}(N_t = 1).\end{aligned}$$

- Case: $i \ge 0, j \ge 2$.

$$\begin{aligned}&\mathrm{P}(N_s = i, N_{s+t} - N_s = j) = \mathrm{P}(J_i \le s < J_{i+1}, J_{i+j} \le s+t < J_{i+j+1})\\ &= \int_{s+t}^\infty \int_s^{s+t} \int_s^{x_3} \int_0^s f_{J_i, J_{i+1}, J_{i+j}, J_{i+j+1}}(x_1, x_2, x_3, x_4)dx_1 dx_2 dx_3 dx_4.\end{aligned}$$

Consider the transformation

$$\begin{aligned}U_1 &= J_i,\\ U_2 &= J_{i+1} - J_i,\\ U_3 &= J_{i+j} - J_{i+1},\\ U_4 &= J_{i+j+1} - J_{i+j}.\end{aligned}$$

Due to the independence of the inter-arrival times the Us are independent of each other and their joint density is given by

$$f_{U_1,U_2,U_3,U_4}(u_1,u_2,u_3,u_4) = f_{\Gamma(i,\lambda)}(u_1) f_{\mathrm{Exp}(\lambda)}(u_2) f_{\Gamma(j-1,\lambda)}(u_3) f_{\mathrm{Exp}(\lambda)}(u_4),$$

where the subscripts denote the associated distributions (Gamma and exponential) with their respective parameters. Using the transformation formula we can find the joint density for the Js. Here we note that the associated Jacobian determinant is equal to 1 since

$$\det \frac{\partial U}{\partial T} = \det \begin{pmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\end{pmatrix} = 1.$$

Then

$$f_{J_i,J_{i+1},J_{i+j},J_{i+j+1}}(x_1,x_2,x_3,x_4) = f_{U_1,U_2,U_3,U_4}(x_1,x_2-x_1,x_3-x_2,x_4-x_3) \times 1$$
$$= f_{\Gamma(i,\lambda)}(x_1) f_{\text{Exp}(\lambda)}(x_2-x_1) f_{\Gamma(j-1,\lambda)}(x_3-x_2) f_{\text{Exp}(\lambda)}(x_4-x_3).$$

Now we need to integrate with respect to all four variables (we omit the steps here since it is a standard (but tedious) calculation), we get

$$\begin{aligned}
&\mathrm{P}(N_s = i, N_{s+t} - N_s = j) \\
&= \int_{s+t}^{\infty}\int_{s}^{s+t}\int_{s}^{x_3}\int_{0}^{s} f_{J_i,J_{i+1},J_{i+j},J_{i+j+1}}(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4 \\
&= \frac{(\lambda s)^i}{i!}e^{-\lambda s}\frac{(\lambda t)^j}{j!}e^{-\lambda t} = \mathrm{P}(N_s = i)\mathrm{P}(N_t = j).
\end{aligned}$$

One can now extend the above proof to cover an arbitrary finite number of increments over disjoint intervals, which will conclude the proof.

3.3 Prerequisites: Lecture 16

Exercise 3- 26: Consider two independent Poisson processes $\{N_t^{(1)}\}_{t\geq 0}$ and $\{N_t^{(2)}\}_{t\geq 0}$ (of rates $\lambda_1 > 0$ and $\lambda_2 > 0$), and we define a new stochastic process

$$N_t = N_t^{(1)} + N_t^{(2)}.$$

Show that $\{N_t\}_{t\geq 0}$ is a Poisson process of rate $\lambda_1 + \lambda_2$.

Solution: Here we seek to verify the properties of a Poisson process for $(N_t)_{t\geq 0}$. It is enough, to consider the second definition of a Poisson process.

First, $N_0 = N_0^{(1)} + N_0^{(2)} = 0$.

Second, let us consider the independence (or not) of the increments. We can argue as follows: Given any choice $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$,

- the random variables $N_{t_0}^{(1)}, N_{t_1}^{(1)} - N_{t_0}^{(1)}, \ldots, N_{t_n}^{(1)} - N_{t_{n-1}}^{(1)}$ are independent since $N^{(1)}$ is a Poisson process,
- the random variables $N_{t_0}^{(2)}, N_{t_1}^{(2)} - N_{t_0}^{(2)}, \ldots, N_{t_n}^{(2)} - N_{t_{n-1}}^{(2)}$ are independent since $N^{(2)}$ is a Poisson process,
- the processes $N^{(1)}$ and $N^{(2)}$ are independent.

We can conclude that also $N_{t_0}^{(1)} + N_{t_0}^{(2)}, N_{t_1}^{(1)} - N_{t_0}^{(1)} + N_{t_1}^{(2)} - N_{t_0}^{(2)}, \ldots, N_{t_n}^{(1)} - N_{t_{n-1}}^{(1)} + N_{t_n}^{(2)} - N_{t_{n-1}}^{(2)}$ are independent.

More formally, one could give a proof by induction.

We sketch the key argument:

$$\mathrm{P}(\cap_{i=1}^{n}\{N_{t_i} - N_{t_{i-1}} = k_i\})$$
$$= \sum_{\nu_l,\mu_l\geq 0,\nu_l+\mu_l=k_l,l=1,\ldots,n}$$

$$\begin{aligned}
&\mathrm{P}((\cap_{i=1}^n \{N_{t_i} - N_{t_{i-1}} = k_i\}) \cap (\cap_{j=1}^n (\{N_{t_j}^{(1)} - N_{t_{j-1}}^{(1)} = \nu_j\} \cap \{N_{t_j}^{(2)} - N_{t_{j-1}}^{(2)} = \mu_j\}))) \\
&= \sum_{\nu_l,\mu_l \geq 0, \nu_l+\mu_l = k_l, l=1,\ldots,n} \mathrm{P}(\cap_{j=1}^n (\{N_{t_j}^{(1)} - N_{t_{j-1}}^{(1)} = \nu_j\} \cap \{N_{t_j}^{(2)} - N_{t_{j-1}}^{(2)} = \mu_j\})) \\
&= \sum_{\nu_l,\mu_l \geq 0, \nu_l+\mu_l = k_l, l=1,\ldots,n} \prod_{j=1}^n \mathrm{P}(\{N_{t_j}^{(1)} - N_{t_{j-1}}^{(1)} = \nu_j\}) \mathrm{P}(\{N_{t_j}^{(2)} - N_{t_{j-1}}^{(2)} = \mu_j\}) \\
&= \sum_{\nu_l,\mu_l \geq 0, \nu_l+\mu_l = k_l, l=1,\ldots,n} \prod_{j=1}^n \mathrm{P}(\{N_{t_j}^{(1)} - N_{t_{j-1}}^{(1)} = \nu_j\} \cap \{N_{t_j}^{(2)} - N_{t_{j-1}}^{(2)} = \mu_j\}) \\
&= \prod_{j=1}^n \mathrm{P}(\{N_{t_j} - N_{t_{j-1}} = k_j\},
\end{aligned}$$

where we have used the independence of $\{N_t^{(1)}\}$ and $\{N_t^{(2)}\}$ and the independence of their increments.

Third, let us consider the last property. For $0 \leq s < t$ we have $N_t - N_s = (N_t^{(1)} - N_s^{(1)}) + (N_t^{(2)} - N_s^{(2)})$, so we simply need to find the distribution of the sum of two independent Poisson random variables. Let us use the Laplace transform with $u > 0$.

$$\begin{aligned}
\mathrm{E}[e^{-u(N_t - N_s)}] &= \mathrm{E}[e^{-u[(N_t^{(1)} - N_s^{(1)}) + (N_t^{(2)} - N_s^{(2)})]}] \\
&= \mathrm{E}[e^{-u(N_t^{(1)} - N_s^{(1)})}]\mathrm{E}[e^{-u(N_t^{(2)} - N_s^{(2)})}],
\end{aligned}$$

where we used the independence of $N^{(1)}$ and $N^{(2)}$.

Using Lemma 5.3.7 and the fact that the increments of $N^{(1)}$ and $N^{(2)}$ follow a Poisson distribution, we get

$$\mathrm{E}[e^{-u(N_t - N_s)}] = \exp\{(\lambda_1 + \lambda_2)(t - s)\{e^{-u} - 1\}\},$$

which is a Poisson random variable of rate $(\lambda_1 + \lambda_2)(t - s)$.

Note that the previous equation also implies the stationarity of the increments.

Exercise 3- 27: Customers arrive at a bank according to a Poisson process $(N_t)_{t\geq 0}$ at a mean rate of $\lambda = 10$ per minute. 60% of the customers wish to withdraw money (type A), 30% wish to pay in money (type B), and 10% wish to do something else.

(a) What is the probability that more than 5 customers arrive in 30 seconds?

(b) What is the probability that in 1 minute, 6 type A customers, 3 type B customers, and 1 type C customers arrive?

(c) If 20 customers arrive in 2 minutes, what is the probability that just one wants to carry out a type C transaction?

(d) What is the probability that the first 3 customers arriving require only to make a type A transaction?

(e) How long a time will elapse until there is a probability of 0.9 that at least one customer of type A and one of type B will have arrived? (You will need to solve this numerically).

Solution:

(a) The number, X, of customers arriving in 30 seconds has a Poisson distribution with parameter $10 \times 0.5 = 5$. (In fact, $X = N_{0.5} \sim \text{Poi}(\lambda \times 0.5) = \text{Poi}(5)$.) So $\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \leq$

$5) = 0.3840$.

(b) Each type of customer arrives according to a Poisson process, independently of the other two types. In particular,

- customers of type A arrive according to a Poisson process $\{N_t^A\}$ of intensity $\lambda \times 0.6 = 6$,
- customers of type B arrive according to a Poisson process $\{N_t^B\}$ of intensity $\lambda \times 0.3 = 3$,
- and customers of type C arrive according to a Poisson process $\{N_t^C\}$ of intensity $\lambda \times 0.1 = 1$,

where the rates are given per minute. This is the thinning of a Poisson processes into three thinned Poisson processes. Thus

$$\begin{aligned} \mathrm{P}(N_1^A = 6)\mathrm{P}(N_1^B = 3)\mathrm{P}(N_1^C = 1) &= 0.1606 \times 0.2240 \times 0.3679 \\ &= 0.0132. \end{aligned}$$

(c) As 20 events are known to have occurred, time is irrelevant. The number X_C of type C customers is distributed as $\text{Binomial}(20, 0.1)$ so that

$$\mathrm{P}(X_C = 1) = \binom{20}{1}(0.1)(0.9)^{19} = 0.270.$$

(d) The probability that any customer requires a type A transaction is 0.6. Hence the probability that the first three customers are type A is $(0.6)^3 = 0.216$.

(e) Suppose that the required time is t. The probability that at least one type A customer has arrived by time t is $(1 - e^{-6t})$ and for type B it is $(1 - e^{-3t})$. Since the two Poisson processes are independent, we must find t such that

$$(1 - e^{-6t})(1 - e^{-3t}) = 0.9.$$

Solving this numerically gives $t = 0.794$ minutes.

Exercise 3- 28: Suppose that cars arrive at the petrol station according to a Poisson process, $\{N_t\}_{t\geq 0}$ of rate $\lambda > 0$. In addition, independently, a car is green with probability p; let $\{N_t^g\}_{t\geq 0}$ denote the number of green cars that have appeared. Show that $\{N_t^g\}_{t\geq 0}$ is a Poisson process.

Solution: Note that this question is about thinning of a Poisson process!

1. Clearly $N_0^g \equiv 0$.
2. Next we consider the independent increments property. We can argue as follows: We know that $\{N_t\}_{t\geq 0}$ is a Poisson process. At each time when an event occurs (i.e. when a car arrives), the event is (independently) classified as "green car" with probability p and as "not a green car" with probability $1 - p$. Accordingly, we can define two independent stochastic processes $\{N_t^g\}_{t\geq 0}$ and $\{N_t^{ng}\}_{t\geq 0}$, where $N_t = N_t^g + N_t^{ng}$ for all $t \geq 0$. Both processes inherit the independent increments from $\{N_t\}_{t\geq 0}$.
3. By similar arguments, we also conclude that the thinned processes inherit the stationarity of the increments from the process $\{N_t\}_{t\geq 0}$ by construction.

4. Finally, we study the law of the number of green cars at time t. Let $k \in \{0, 1, 2, \dots\}$. Then

$$\begin{aligned}
\mathrm{P}(N_t^g = k) &= \sum_{n=k}^{\infty} \mathrm{P}(N_t^g = k, N_t = n) \\
&= \sum_{n=k}^{\infty} \mathrm{P}(N_t^g = k | N_t = n)\mathrm{P}(N_t = n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \mathrm{P}(N_t = n) \\
&= e^{-\lambda t} \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(\lambda t)^n}{n!} \\
&= \left(\frac{p}{1-p}\right)^k \frac{e^{-\lambda t}}{k!} \sum_{n=k}^{\infty} \left(\frac{[(1-p)\lambda t]^n}{(n-k)!}\right) \\
&= \left(\frac{p}{1-p}\right)^k \frac{e^{-\lambda t}}{k!} [(1-p)\lambda t]^k \sum_{s=0}^{\infty} \left(\frac{[(1-p)\lambda t]^s}{s!}\right) \\
&= \frac{(\lambda p t)^k}{k!} e^{-\lambda p t}
\end{aligned}$$

which completes the solution.

Exercise 3- 29: A bank opens at 10.00am and customers arrive according to a non-homogeneous Poisson process at a rate $10(1+2t)$, measured in hours, starting from 10.00.

(a) What is the probability that two customers have arrived by 10.05?

(b) What is the probability that 6 customers arrive between 10.45 and 11.00?

(c) What is the probability that more than 50 customers arrive between 11.00 and 12.00?

(d) What is the median time to the first arrival after the bank opens?

(e) By what time is there a probability of 0.95 that the first customer after 11.00 will have arrived?

Solution: Here the intensity function is given by $\lambda(t) = 10(1+2t)$. We denote by $N = (N_t)$ the inhomogeneous Poisson process modelling the arrivals of the customers, time t is measured in hours, and $t = 0$ is associated with the opening of the bank at time 10am.

(a) We first compute the integrated intensity function

$$m(t) = \int_0^t \lambda(u)du = \int_0^t 10(1+2u)du = 10(t+t^2).$$

Five minutes is 1/12 hours, so that the number of customers arriving between 10.00 and 10.05 is a Poisson process with rate m(1/12) = 0.9028. It follows that the probability that two customers have arrived by 10.05 is

$$\mathrm{P}(N_{1/12} = 2) = \frac{m(1/12)^2}{2} \exp(-m(1/12)) = 0.1652.$$

(b) Converting to the units we need, we want to know the probability that 6 customers arrive in the interval $[0.75, 1.00]$. We know that $N_1 - N_{0.75}$ will have a Poisson distribution with parameter $m(1) - m(0.75) = 6.875$. From which it follows that the probability that 6 customers will arrive is

$$P(N_1 - N_{0.75} = 6) = \frac{(6.875)^6}{6!} \exp(-6.875) = 0.1515.$$

(c) The number of customers arriving between these times, given by $N_2 - N_1$, will have a Poisson distribution with parameter $m(2) - m(1) = 40$. The probability that more than 50 arrive can be obtained by summing the relevant probabilities from a table of the Poisson distribution, with parameter 40, or by using a calculator. Then we find that

$$P(N_2 - N_1 > 50) = 0.05262805.$$

(d) Let X_1 denote the first inter-arrival time, i.e. the time between the opening of the bank (at time $t = 0$) and the time the first customer arrives. We deduce that, for $t > 0$,

$$P(X_1 > t) = P(N_t = 0) = \exp(-m(t)).$$

Hence

$$F_{X_1}(t) = P(X_1 \leq t) = 1 - \exp(-m(t)) = 1 - \exp(-10(t + t^2)).$$

The median of this is given by the value of $m > 0$ for which

$$P(X_1 \leq m) = 1 - \exp(-10(m + m^2)) = 0.5.$$

We note that the above quadratic equation has two roots: $m_1 = 0.065, m_2 = -1.065$. We take the positive root, yielding $m = 0.065$, so that the median time of arrival of the first customer is about 3.9 minutes after 10.

(e) Let X_2 denote the first inter-arrival time after 11am, i.e. it denotes the length of time between 11am and the arrival of the first customer after 11am. Note that the time 11am corresponds to $t = 1$. Let $x > 0$, then

$$P(X_2 > x) = P(N_{1+x} - N_1 = 0) = \exp(-(m(1 + x) - m(1))).$$

We note that

$$m(1 + x) - m(1) = 10x^2 + 30x.$$

Hence, the corresponding cdf is given by

$$F_{X_2}(x) = P(X_2 \leq x) = 1 - \exp(-(m(1 + x) - m(1))) = 1 - \exp(-10x^2 - 30x).$$

We need to find $x > 0$ such that

$$0.95 = F_{X_2}(x) = 1 - \exp(-10x^2 - 30x).$$

Solving for x, we obtain two roots:

$$x_{1/2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{\log(0.05)}{10}},$$

i.e. $x_1 = 0.09673831$ and $x_2 = -3.096738$. Again, we take the positive root $x \approx 0.097$, which corresponds to 5.8 minutes. I.e. there is a 0.95 probability, that at least one customer will have arrived between 11am and 11:06am.

Alternative solution: You could also solve this problem slightly differently: Suppose T denotes the time of the arrival of the first customer after 11am, i.e. we assume that $T > 1$. Then we want to find the cdf of T. We note that for $t > 1$, we have that

$$\mathrm{P}(T > t) = \mathrm{P}(N_t - N_1 = 0) = \exp(-(m(t) - m(1))) = \exp(-10t - 10t^2 + 20).$$

Hence,

$$F_T(t) = 1 - \exp(-10t - 10t^2 + 20).$$

Now we need to find $t > 1$ such that

$$0.95 = F_T(t) = 1 - \exp(-10t - 10t^2 + 20).$$

When we solve this equation for t, we get two roots:

$$t_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{9}{4} - \frac{\log(0.05)}{10}},$$

i.e. $t_1 = 1.096738$ and $t_2 = -2.096738$. Taking the positive root, which is greater than 1, we get the same result as above. I.e. there is a probability of 0.95 that at least one customer will have arrived after 11.00 and before 11:06.

3.4 Prerequisites: Lecture 17

Exercise 3- 30: Let $\{N_t\}$ be a Poisson process of rate $\lambda > 0$ and $Y, Y_1, Y_2, \dots$ be a sequence of i.i.d random variables, such that their characteristic function exists. Further $\{N_t\}$ and $\{Y_j\}$ are independent. Let

$$S_t = \sum_{j=1}^{N_t} Y_j.$$

Find the characteristic function of S_t, $t > 0$.

Solution: For $u \in \mathbb{R}$, let $\psi_{S_t}(u)$ (resp. $\psi_Y(u)$) denote the characteristic function of S_t (resp. Y). We use the law of total expectation:

$$\psi_{S_t}(u) = \mathrm{E}[\exp(iuS_t)] = \mathrm{E}\left[\exp\left(iu\sum_{j=1}^{N_t} Y_j\right)\right] = \mathrm{E}\left[\mathrm{E}\left[\exp\left(iu\sum_{j=1}^{N_t} Y_j\right)\middle| N_t\right]\right],$$

where

$$\mathrm{E}\left[\exp\left(iu\sum_{j=1}^{N_t} Y_j\right)\middle| N_t = n\right] = \mathrm{E}\left[\exp\left(iu\sum_{j=1}^{n} Y_j\right)\middle| N_t = n\right] \overset{(N_t),(Y_j)\text{ indep.}}{=} \mathrm{E}\left[\exp\left(iu\sum_{j=1}^{n} Y_j\right)\right]$$

$$\stackrel{(Y_j)\text{ i.i.d.}}{=} \{\mathrm{E}\left[\exp\left(iuY\right)\right]\}^n = (\psi_Y(u))^n.$$

Hence,

$$\begin{aligned}\psi_{S_t}(u) &= \sum_{n=0}^{\infty} \mathrm{E}\left[\exp\left(iu\sum_{j=1}^{N_t} Y_j\right)\middle| N_t = n\right] \mathrm{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} (\psi_Y(u))^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t \psi_Y(u))^n}{n!} \\ &= e^{-\lambda t} e^{\lambda t \psi_Y(u)} = \exp\{\lambda t[\psi_Y(u) - 1]\}.\end{aligned}$$

Exercise 3- 31: A person makes shopping expeditions according to a Poisson process with rate $\lambda > 0$. The number of purchases he makes during each expedition is distributed according to a geometric distribution $\text{Geometric}(p)$. What are the mean and variance of the total number of purchases made in time t? *Hint: Please use the following probability mass function of the geometric distribution* $p(y) = p(1-p)^{y-1}, \quad y = 1, 2 \ldots.$

Solution: We consider the compound Poisson process

$$S_t = \sum_{j=1}^{N_t} Y_j,$$

where the jump sizes have geometric distribution

$$\mathrm{P}(Y_j = y) = p(1-p)^{y-1}, \quad y = 1, 2 \ldots.$$

From Theorem 5.5.13 we know that for $t \geq 0$,

$$\mathrm{E}(S_t) = \lambda t \mathrm{E}(Y_1), \qquad \mathrm{Var}(S_t) = \lambda t \mathrm{E}(Y_1^2).$$

Hence, we only need to compute the mean and variance of the geometric distribution. We compute the corresponding moment generating function:

$$\begin{aligned}M_{Y_1}(u) = \mathrm{E}(e^{uY_1}) = \sum_{y=1}^{\infty} e^{uy}\mathrm{P}(Y_1 = y) = \sum_{y=1}^{\infty} e^{uy} p(1-p)^{y-1} = \frac{p}{1-p}\sum_{y=1}^{\infty}(e^u(1-p))^y \\ = \frac{p}{1-p}\left((1 - e^u(1-p))^{-1} - 1\right) = \frac{p}{1-p} \times \frac{e^u(1-p)}{1 - e^u(1-p)} \\ = \frac{pe^u}{1 - e^u(1-p)} = p(e^{-u} - 1 + p)^{-1},\end{aligned}$$

which is well-defined for all u such that $e^u(1-p) < 1 \Leftrightarrow u < -\log(1-p)$. Differentiating leads to

$$\begin{aligned}M'_{Y_1}(u) &= p(e^{-u} - 1 + p)^{-2} e^{-u}, \\ M''_{Y_1}(u) &= 2p(e^{-u} - 1 + p)^{-3} e^{-2u} - p(e^{-u} - 1 + p)^{-2} e^{-u}.\end{aligned}$$

Hence

$$\mathrm{E}(Y_1) = M'_{Y_1}(0) = \frac{1}{p}, \quad \mathrm{E}(Y_1^2) = M''_{Y_1}(0) = \frac{2-p}{p^2}, \qquad [\mathrm{Var}(Y_1) = \frac{1-p}{p^2}].$$

Hence, we conclude that

$$\mathrm{E}(S_t) = \frac{\lambda t}{p}, \qquad \mathrm{Var}(S_t) = \frac{\lambda t(2-p)}{p^2}.$$

Note that if you use the alternative form of the geometric distribution with probability mass function given by

$$p(y) = p(1-p)^y \quad y = 0, 1 \ldots,$$

then you will get a different answer!

Exercise 3- 32: A physicist has a large lump of radioactive material of very long half- life. She records radioactive decays from this lump using a suitable counter. The counter is switched on at 12.00 noon and then left running. Let N_t be the total number of decays recorded by the counter after it has been switched on for t hours.

(a) Let D_t be the total number of decays which occur in the radioactive material in the period of t hours starting at noon. Suppose that $\{D_t\}_{t\geq 0}$ is a Poisson process of rate $\mu > 0$. Each decay is recorded by the counter with probability p independently of whether other decays are recorded and independently of when decays occur. What is the distribution of N_t? (Hint: Consider the probability generating function).

(b) Suppose $\{N_t\}_{t\geq 0}$ is a Poisson process of rate $\lambda = 3$ per hour. Having switched the counter on at 12 noon, the physicist goes to lunch. She returns at 1pm. What is the probability that the counter has recorded exactly one decay by this time? Given that exactly one decay has occurred, show that the probability that this decay occurred after 12.45pm is 0.25.

Solution:

(a) Suppose that

$$W_i = \begin{cases} 1 & \text{if the} \quad i^{th} \quad \text{decay is recorded} \\ 0 & \text{if the} \quad i^{th} \quad \text{decay isn't recorded} \end{cases}$$

Then $N_t = \sum_{i=1}^{D_t} W_i$. Since the W_i's are independent and D_t is independent of the W_i's we have, for $|s| < 1$,

$$\begin{aligned} G_{N_t}(s) &= \mathrm{E}(s^{N_t}) = \mathrm{E}\left(s^{\sum_{i=1}^{D_t} W_i}\right) = \mathrm{E}\left(\mathrm{E}\left(s^{\sum_{i=1}^{D_t} W_i} \middle| D_t\right)\right) \\ &= \mathrm{E}\left(\mathrm{E}\left(\prod_{i=1}^{D_t} s^{W_i} \middle| D_t\right)\right) = \mathrm{E}\left(\prod_{i=1}^{D_t} \mathrm{E}\left(s^{W_i}\right)\right) \\ &= \mathrm{E}\left(\prod_{i=1}^{D_t} G_{W_i}(s)\right) = \mathrm{E}\left((G_W(s))^{D_t}\right) = G_{D_t}(G_W(s)). \end{aligned}$$

Since $D_t \sim \mathrm{Poi}(\mu t)$, $G_{D_t}(s) = e^{\mu t(s-1)}$ and $G_W(s) = (1-p) + ps$ it follows

$$G_{N_t}(s) = e^{\mu p t(s-1)},$$

i.e. $N_t \sim \text{Poi}(\mu p t)$.

Alternatively, you could argue that this is nothing else than the thinning of a Poisson process!

(b) $\text{P}(N_1 = 1) = 3e^{-3}$. Now

$$\begin{aligned}
\text{P}(J_1 > 0.75 | N_1 = 1) &= \frac{\text{P}(J_1 > 0.75, N_1 = 1)}{\text{P}(N_1 = 1)} \\
&= \frac{\text{P}(\text{no event in}[0, 0.75], \text{one event in}[0.75, 1])}{\text{P}(N_1 = 1)} \\
&= \frac{\text{P}(\text{no event in}[0, 0.75])\text{P}(\text{one event in}[0.75, 1])}{\text{P}(N_1 = 1)} \\
&= \frac{\text{P}(N_{\frac{3}{4}} = 0)\text{P}(N_1 - N_{\frac{3}{4}} = 1)}{\text{P}(N_1 = 1)}
\end{aligned}$$

where we have used independent increments. Substituting the relevant probability mass functions (using $N_t \sim \text{Poi}(3t)$) completes the exercise:

$$\text{P}(J_1 > 0.75 | N_1 = 1) = \frac{e^{-\frac{3}{4} \times 3} \times e^{-\frac{1}{4} \times 3} \frac{1}{4} \times 3}{e^{-3} \times 3} = \frac{1}{4}.$$

Alternatively, you could argue that $J_1 | N_1 = 1$ follows a uniform distribution on $[0, 1]$, from which the result follows.