4 Problem sheet 4: Continuous-time Markov chains

4.1 Prerequisites: Lecture 18

Exercise 4- 33: Let $N = (N_t)_{t\geq 0}$ denote a Poisson process with rate $\lambda > 0$. Define a stochastic process $Z = (Z_t)_{t\geq 0}$ with $Z_t = (-1)^{N_t}$.

(a) Determine the state space E_Z of all possible values Z can take.

(b) Sketch a sample path of the process Z and describe how long on average you need to wait until the process switches between different values in E_Z.

(c) Find the probability mass function of Z_t for $t \geq 0$.

(d) Find $\mathrm{E}(Z_t)$ for $t \geq 0$.

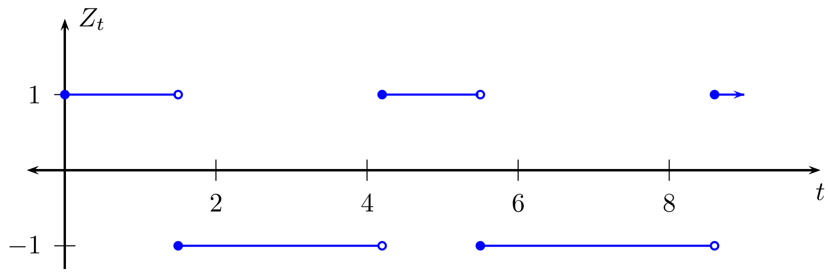
(e) Find $\mathrm{P}(Z_s = Z_t)$ for $0 \leq s < t$.

(f) Determine whether Z is a continuous-time Markov chain and prove/justify your answer carefully.

Solution:

(a) Z can only take the values 1 and -1, hence $E_Z = \{-1, 1\}$.

(b) Since $N_0 = 0$, we have that $Z_0 = 1$. So Z starts at level 1 and then switches between the values 1 and -1. One possible sample path of Z is given by



Since the inter-arrival times of a Poisson process of rate λ follow the exponential distribution with parameter λ, which has mean $1/\lambda$, we have to wait on average $1/\lambda$ until we see a switch in the values of N. As soon as N changes value, the value of Z will also change, hence we have to wait on average $1/\lambda$ until we see a switch in the values of Z.

(c) Let $t \geq 0$, then

$$\mathrm{P}(Z_t = 1) = \mathrm{P}((-1)^{N_t} = 1) = \mathrm{P}(N_t \text{ is even}) = \sum_{k=0}^{\infty} \mathrm{P}(N_t = 2k)$$
$$= \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k}}{(2k)!} e^{-\lambda t} = \frac{1}{2}(e^{\lambda t} + e^{-\lambda t})e^{-\lambda t} = \frac{1}{2}(1 + e^{-2\lambda t}),$$
$$\mathrm{P}(Z_t = -1) = \mathrm{P}((-1)^{N_t} = -1) = \mathrm{P}(N_t \text{ is odd}) = \sum_{k=0}^{\infty} \mathrm{P}(N_t = 2k+1)$$
$$= \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(2k+1)!} e^{-\lambda t} = \frac{1}{2}(e^{\lambda t} - e^{-\lambda t})e^{-\lambda t} = \frac{1}{2}(1 - e^{-2\lambda t}),$$

and $\mathrm{P}(Z_t = x) = 0$ for $x \notin E_Z$.

Note that we used that

$$e^x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!},$$

and

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{1}{2}(e^x + e^{-x})[= \cosh(x)], \quad \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{2}(e^x - e^{-x})[= \sinh(x)].$$

(d) For $t \geq 0$, we have

$$\mathrm{E}(Z_t) = 1 \times \mathrm{P}(Z_t = 1) + (-1) \times \mathrm{P}(Z_t = -1) = \mathrm{P}(Z_t = 1) - \mathrm{P}(Z_t = -1) = e^{-2\lambda t}.$$

(e) Let $0 \leq s < t$. Then, using that the Poisson process has stationary increment, we get

$$\begin{aligned}
\mathrm{P}(Z_s = Z_t) &= \mathrm{P}(Z_s - Z_t = 0) = \mathrm{P}((-1)^{N_s} - (-1)^{N_t} = 0) \\
&= \mathrm{P}((-1)^{N_s}[1 - (-1)^{N_t - N_s}] = 0) = \mathrm{P}(1 - (-1)^{N_t - N_s} = 0) \\
&= \mathrm{P}(1 = (-1)^{N_t - N_s}) = \mathrm{P}(N_t - N_s \text{ is even}) \\
&\overset{\text{stat. incr.}}{=} \mathrm{P}(N_{t-s} \text{ is even}) \overset{(c)}{=} \frac{1}{2}\left(1 + e^{-2\lambda(t-s)}\right).
\end{aligned}$$

(f) Yes, Z is a continuous time Markov chain on the state space E_Z.

Proof: Z is a continuous-time stochastic process taking values in E_Z. It remains to prove the Markov property. Consider any sequence $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ for any $n \in \mathbb{N}$ and any states $i_1, \ldots, i_n \in E_Z$, then

$$\begin{aligned}
A &:= \mathrm{P}(Z_{t_n} = i_n | Z_{t_{n-1}} = i_{n-1}) = \frac{\mathrm{P}(Z_{t_n} = i_n, Z_{t_{n-1}} = i_{n-1})}{\mathrm{P}(Z_{t_{n-1}} = i_{n-1})} \\
&= \frac{\mathrm{P}\left(\frac{Z_{t_n}}{Z_{t_{n-1}}} = \frac{i_n}{i_{n-1}}, Z_{t_{n-1}} = i_{n-1}\right)}{\mathrm{P}(Z_{t_{n-1}} = i_{n-1})} = \frac{\mathrm{P}\left((-1)^{N_{t_n} - N_{t_{n-1}}} = \frac{i_n}{i_{n-1}}, Z_{t_{n-1}} = i_{n-1}\right)}{\mathrm{P}(Z_{t_{n-1}} = i_{n-1})} \\
&= \frac{\mathrm{P}\left((-1)^{N_{t_n} - N_{t_{n-1}}} = \frac{i_n}{i_{n-1}}\right)\mathrm{P}\left(Z_{t_{n-1}} = i_{n-1}\right)}{\mathrm{P}(Z_{t_{n-1}} = i_{n-1})} = \mathrm{P}\left((-1)^{N_{t_n} - N_{t_{n-1}}} = \frac{i_n}{i_{n-1}}\right),
\end{aligned}$$

where we used that the increments of a Poisson process are independent, hence $N_{t_n} - N_{t_{n-1}}$ and $N_{t_{n-1}}$ are independent, which implies that $N_{t_n} - N_{t_{n-1}}$ and $(-1)^{N_{t_{n-1}}}$ are independent (since the latter is just a function of $N_{t_{n-1}}$).

A similar argument can be applied to the general case. Here we note that the set of equations

$$Z_{t_n} = i_n, Z_{t_{n-1}} = i_{n-1}, \ldots, Z_{t_1} = i_1,$$

and

$$\frac{Z_{t_n}}{Z_{t_{n-1}}} = \frac{i_n}{i_{n-1}}, \frac{Z_{t_{n-1}}}{Z_{t_{n-2}}} = \frac{i_{n-1}}{i_{n-2}}, \ldots, \frac{Z_{t_2}}{Z_{t_1}} = \frac{i_2}{i_1}, Z_{t_1} = i_1, \tag{4.1}$$

are equivalent. Equation (4.1) is equivalent to

$$(-1)^{N_{t_n} - N_{t_{n-1}}} = \frac{i_n}{i_{n-1}}, (-1)^{N_{t_{n-1}} - N_{t_{n-2}}} = \frac{i_{n-1}}{i_{n-2}}, \ldots, (-1)^{N_{t_2} - N_{t_1}} = \frac{i_2}{i_1}, (-1)^{N_{t_1}} = i_1,$$

which on the respective left hand sides are functions of the independent increments of the Poisson process. Hence we get

$$
\begin{aligned}
B &:= \mathrm{P}(Z_{t_n} = i_n | Z_{t_{n-1}} = i_{n-1}, \dots, Z_{t_1} = i_1) \\
&= \frac{\mathrm{P}(Z_{t_n} = i_n, Z_{t_{n-1}} = i_{n-1}, \dots, Z_{t_1} = i_1)}{\mathrm{P}(Z_{t_{n-1}} = i_{n-1}, \dots, Z_{t_1} = i_1)} =: \frac{C}{D},
\end{aligned}
$$

where the numerator is given by

$$
\begin{aligned}
C &= \mathrm{P}((-1)^{N_{t_n} - N_{t_{n-1}}} = \frac{i_n}{i_{n-1}}, (-1)^{N_{t_{n-1}} - N_{t_{n-2}}} = \frac{i_{n-1}}{i_{n-2}}, \dots, (-1)^{N_{t_2} - N_{t_1}} = \frac{i_2}{i_1}, (-1)^{N_{t_1}} = i_1) \\
&= \mathrm{P}((-1)^{N_{t_n} - N_{t_{n-1}}} = \frac{i_n}{i_{n-1}}) \times \mathrm{P}((-1)^{N_{t_{n-1}} - N_{t_{n-2}}} = \frac{i_{n-1}}{i_{n-2}}) \times \cdots \\
&\quad \times \mathrm{P}((-1)^{N_{t_2} - N_{t_1}} = \frac{i_2}{i_1}) \times \mathrm{P}((-1)^{N_{t_1}} = i_1),
\end{aligned}
$$

and the denominator by

$$
\begin{aligned}
D &= \mathrm{P}((-1)^{N_{t_{n-1}} - N_{t_{n-2}}} = \frac{i_{n-1}}{i_{n-2}}, \dots, (-1)^{N_{t_2} - N_{t_1}} = \frac{i_2}{i_1}, (-1)^{N_{t_1}} = i_1) \\
&= \mathrm{P}((-1)^{N_{t_{n-1}} - N_{t_{n-2}}} = \frac{i_{n-1}}{i_{n-2}}) \times \cdots \times \mathrm{P}((-1)^{N_{t_2} - N_{t_1}} = \frac{i_2}{i_1}) \times \mathrm{P}((-1)^{N_{t_1}} = i_1).
\end{aligned}
$$

Altogether, we have that $B = C/D = A$, which concludes the proof.

Exercise 4- 34: A machine can be in one of two states: working or being repaired. When it is in the ”working” state it functions for a time that is exponentially distributed (parameter $\lambda > 0$) before switching to the ”being repaired” state. When it is in the ”being repaired” state it functions for a time that is exponentially distributed (parameter $\nu > 0$) before switching to the ”working” state. We assume independence between the corresponding holding times.

(a) Given that the machine starts in the ”working” state, what is the mean time until:

- it breaks down for the first time?
- it breaks down for the third time?

(b) What is the variance of the time until

- it breaks down for the first time?
- it breaks down for the third time?

Solution: Let $X_i \sim \mathrm{Exp}(\lambda)$ be the length of time the machine functions for the ith time before it breaks down again (for $i = 1, 2, \dots$). Let $Y_i \sim \mathrm{Exp}(\nu)$ denote the length of time the machine is in the being repaired state for the ith time (for $i = 1, 2, \dots$). All X_i and Y_i are independent.

(a) Then

$$
\mathrm{E}(X_1) = \frac{1}{\lambda}, \quad \mathrm{E}(X_1 + X_2 + X_3 + Y_1 + Y_2) = \frac{3}{\lambda} + \frac{2}{\nu}.
$$

(b)

$$
\mathrm{Var}(X_1) = \frac{1}{\lambda^2}, \quad \mathrm{Var}(X_1 + X_2 + X_3 + Y_1 + Y_2) = \frac{3}{\lambda^2} + \frac{2}{\nu^2}.
$$

Exercise 4- 35: The following question is adapted from Grimmett & Stirzaker (2001*b*,*a*), Problem 6.15.14.

Let X be a continuous-time Markov chain with countable state space E and standard semigroup $\mathbf{P}_t$.

(a) Show that $p_{ij}(t)$ is a continuous function of t. *Hint:* Use the Chapman-Kolmogorov equations.

(b) Next, let $g(t) = -\log(p_{ii}(t))$. Show that 1) g is a continuous function, 2) $g(0) = 0$, and 3) g is subadditive, i.e. $g(s+t) \leq g(s) + g(t)$ for all $s, t \geq 0$. From a result from analysis (which you do not need to prove) you may then conclude that

$$\lim_{t \downarrow 0} \frac{g(t)}{t} = \lambda \quad \text{exists and } \lambda = \sup_{t>0} \frac{g(t)}{t} \leq \infty.$$

Deduce that $g_{ii} = \lim_{t\downarrow 0} t^{-1}(p_{ii}(t) - 1)$ exists, but may be equal to ∞.

Solution:

(a) We prove the continuity of $p_{ij}(t)$: Using the Chapman-Kolmogorov equations, we get that, for $t \geq 0$, $h > 0$,

$$\begin{aligned}
|p_{ij}(t+h) - p_{ij}(t)| &= \left| \sum_{k\in E} p_{ik}(h)p_{kj}(t) - \sum_{k\in E} p_{ik}(0)p_{kj}(t) \right| \\
&= \left| \sum_{k\in E} (p_{ik}(h) - \delta_{ik})p_{kj}(t) \right| \\
&= \left| (p_{ii}(h) - 1)p_{ij}(t) + \sum_{k\in E, k\neq i} p_{ik}(h)p_{kj}(t) \right| \\
&\leq (1 - p_{ii}(h))p_{ij}(t) + \sum_{k\in E, k\neq i} p_{ik}(h) \\
&\leq (1 - p_{ii}(h)) + (1 - p_{ii}(h)) \to 0,
\end{aligned}$$

as $h \downarrow 0$, if the semigroup is standard. Also, by the same arguments, for $0 < h < t$,

$$\begin{aligned}
|p_{ij}(t) - p_{ij}(t-h)| &= \left| \sum_{k\in E} p_{ik}(h)p_{kj}(t-h) - \sum_{k\in E} p_{ik}(0)p_{kj}(t-h) \right| \\
&= \left| \sum_{k\in E} (p_{ik}(h) - \delta_{ik})p_{kj}(t-h) \right| \\
&= \left| (p_{ii}(h) - 1)p_{ij}(t-h) + \sum_{k\in E, k\neq i} p_{ik}(h)p_{kj}(t-h) \right| \\
&\leq (1 - p_{ii}(h))p_{ij}(t-h) + \sum_{k\in E, k\neq i} p_{ik}(h) \\
&\leq (1 - p_{ii}(h)) + (1 - p_{ii}(h)) \to 0,
\end{aligned}$$

as $h \downarrow 0$, if the semigroup is standard.

(b) Next, define $g(t) = -\log(p_{ii}(t))$ for $i \in E$.

1) We know that the function $\log(x)$ is continuous for $0 < x \leq 1$; since we have already shown that p_{ii} is a continuous function, we deduce that g is continuous.

2) $g(0) = -\log(p_{ii}(0)) = -\log(1) = 0$.

3) Let $s, t \geq 0$. Then

$$p_{ii}(t+s) \stackrel{CK}{=} \sum_{k\in E} p_{ik}(s)p_{ki}(t) \geq p_{ii}(s)p_{ii}(t).$$

Hence

$$g(s+t) = -\log(p_{ii}(t+s)) \leq -\log(p_{ii}(s)p_{ii}(t)) = g(s) + g(t).$$

From a result from analysis, we conclude that

$$\lim_{t\downarrow 0} \frac{g(t)}{t} = \lambda \quad \text{exists and } \lambda = \sup_{t>0} \frac{g(t)}{t} \leq \infty.$$

Then

$$\lim_{t\downarrow 0} \frac{(p_{ii}(t)-1)}{t} = \lim_{t\downarrow 0} \frac{(p_{ii}(t)-1)}{t}\frac{g(t)}{g(t)} = \lim_{t\downarrow 0} \frac{g(t)}{t}\frac{(p_{ii}(t)-1)}{g(t)}.$$

We note that the fist factor converges to λ as $t \downarrow 0$. For the second factor, we have

$$\frac{(p_{ii}(t)-1)}{g(t)} = \frac{(p_{ii}(t)-1)}{-\log(p_{ii}(t))} = \frac{-(1-p_{ii}(t))}{-\log(1-(1-p_{ii}(t)))} \to -1, \quad \text{as} \quad t \downarrow 0,$$

since

$$\lim_{x\downarrow 0} \frac{x}{\log(1-x)} \stackrel{\mathrm{L'Hospital}}{=} \lim_{x\downarrow 0} \frac{1}{(1-x)^{-1}(-1)} = -1.$$

Hence

$$\lim_{t\downarrow 0} \frac{(p_{ii}(t)-1)}{t} = -\lambda = g_{ii}$$

exists, but may be equal to $-\infty$.

4.2 Prerequisites: Lecture 19

Exercise 4- 36: In this question we will study recurrence and transience for continuous-time Markov chains. We first introduce a definition and state an important result.

Let $X = (X_t)_{t\geq 0}$ be a minimal continuous-time Markov chain on a countable state space E with generator $\mathbf{G}$. We say that state $i \in E$ is *recurrent* if

$$\mathrm{P}(\{t \geq 0 : X_t = i\} \text{ is unbounded } | X_0 = i) = 1.$$

We say that state $i \in E$ is *transient* if

$$\mathrm{P}(\{t \geq 0 : X_t = i\} \text{ is unbounded } | X_0 = i) = 0.$$

We state the following results without proof:

- If state i is recurrent for the jump chain, then i is recurrent for X.
- If state i is transient for the jump chain, then i is transient for X.
- Every state is either recurrent or transient.
- Recurrence and transience are class properties.

See Norris (1998) p. 115 for a proof of the above result.

Let $X = (X_t)_{t\geq 0}$ be a minimal continuous-time Markov chain on a countable state space E with generator $\mathbf{G}$. Suppose $E = \{1, 2, 3, 4\}$ and

$$\mathbf{G} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{6} & 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For each state in the state space, decide whether it is recurrent or transient and justify your answer.

Solution: We derive the transition matrix of the corresponding jump chain:

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We observe that the jump chain has two communicating classes: $T = \{1, 2, 3\}, C = \{4\}$. T is not closed, hence all states in T are transient. C is finite and closed, hence state 4 is (positive) recurrent.

We know that if a state is recurrent (transient) for the jump chain, then it is recurrent (transient) for the continuous-time Markov chain. So we conclude that states 1, 2, 3 are transient and state 4 is recurrent for the continuous-time Markov chain. (Moreover, since $g_{44} = 0$, we have that state 4 is positive recurrent.)

4.3 Prerequisites: Lecture 20

Exercise 4- 37: Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov chain on the state space $E = \{1, 2\}$ with generator

$$\mathbf{G} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

(a) Find the stationary distribution of X.

(b) Find the stationary distribution of the jump chain associated with X.

(c) Find the transition matrix $\mathbf{P}_t = (p_{ij}(t))_{i,j\in E}$ for all $t \geq 0$.
Hint: You may use without a proof that $\mathbf{G} = \mathbf{ODO}^{-1}$, where

$$\mathbf{O} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \mathbf{O}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

Solution:

(a) We denote the stationary distribution of X by $\boldsymbol{\pi} = (\pi_1, \pi_2)$. We solve $\boldsymbol{\pi}\mathbf{G} = \mathbf{0}$, for $\pi_1, \pi_2 \geq 0, \pi_1 + \pi_2 = 1$. Here we have $-\pi_1 + 2\pi_2 = 0 \Leftrightarrow \pi_1 = 2\pi_2$. Then $1 = \pi_1 + \pi_2 \Leftrightarrow 1 = 3\pi_2 \Leftrightarrow \pi_1 = \frac{2}{3}, \pi_2 = \frac{1}{3}$.
(One can show that, since $\mathbf{G}$ is irreducible and recurrent, we get that $\boldsymbol{\pi}\mathbf{G} = \mathbf{0} \Leftrightarrow \boldsymbol{\pi}\mathbf{P}_t = \boldsymbol{\pi}$ for all $t \geq 0$, where $(\mathbf{P}_t)_{t\geq 0}$ denotes the matrix of transition probabilities associated with X.)

(b) The transition matrix of the associated jump chain is given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote the stationary distribution of the jump chain by $\boldsymbol{\pi} = (\pi_1, \pi_2)$. We solve $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$, for $\pi_1, \pi_2 \geq 0, \pi_1 + \pi_2 = 1$. Here we have $\pi_1 = \pi_2$. Then $1 = \pi_1 + \pi_2 \Leftrightarrow \pi_1 = \frac{1}{2}, \pi_2 = \frac{1}{2}$.

(c) From lectures, we know that, for $t \geq 0$, we have

$$\mathbf{P}_t = e^{t\mathbf{G}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{O}\mathbf{D}^n\mathbf{O}^{-1} = \mathbf{O}\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{D}^n\mathbf{O}^{-1}.$$

Hence

$$\mathbf{P}_t = \mathbf{O}\begin{pmatrix} e^{t\cdot 0} & 0 \\ 0 & e^{-3t} \end{pmatrix}\mathbf{O}^{-1} = \begin{pmatrix} \frac{2}{3} + \frac{1}{3}e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ \frac{2}{3} - \frac{2}{3}e^{-3t} & \frac{1}{3} + \frac{2}{3}e^{-3t} \end{pmatrix}.$$

4.4 Prerequisites: Lecture 21

Exercise 4- 38: Let $\{N_t\}_{t\geq 0}$ be a birth process with intensities $\lambda_0, \lambda_1, \ldots$, such that $\lambda_i \neq \lambda_j$ for any $i \neq j$, and $N_0 = 0$. Derive the forward equations for this process. Hence verify that

$$p_n(t) = \frac{1}{\lambda_n}\sum_{i=0}^{n} \lambda_i e^{-\lambda_i t}\left[\prod_{j=0, j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i}\right]$$

where $p_n(t) = \mathrm{P}(N_t = n)$ and the convention $\prod_{\emptyset} = 1$ is used.

Solution: First we derive the forward equations. Let $n \in \mathbb{N}_0$ and define $p_{-1}(t) \equiv 0$. Let $t \geq 0, \delta > 0$. Then

$$\begin{aligned} p_n(t+\delta) = \mathrm{P}(N_{t+\delta} = n) &\overset{\text{LTP}}{=} \sum_{k=0}^{n} \mathrm{P}(N_{t+\delta} = n | N_t = k)\mathrm{P}(N_t = k) \\ &\overset{\text{single arrival}}{=} (1 - \lambda_n\delta)p_n(t) + \lambda_{n-1}\delta p_{n-1}(t) + o(\delta). \end{aligned}$$

Using the usual arguments it follows that

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1}p_{n-1}(t).$$

For $n = 0$ it is clear that

$$p_0'(t) = -\lambda_0 p_0(t)$$

i.e. $p_0(t) = \exp(-\lambda_0 t)$, where we have used the boundary condition $p_0(0) = 1$.

To complete the exercise, we need to verify that $p_n(t)$ as given, is a solution of the forward equations. Since $n = 0$ is clear, we consider $n \in \mathbb{N}$. Now

$$p_n'(t) = -\frac{1}{\lambda_n}\sum_{i=0}^{n} \lambda_i^2 e^{-\lambda_i t}\left[\prod_{j=0, j\neq i}^{n} \frac{\lambda_j}{\lambda_j - \lambda_i}\right]. \quad (4.2)$$

In addition

$$-\lambda_n p_n(t) = -\sum_{i=0}^{n} \lambda_i e^{-\lambda_i t}\left[\prod_{j=0,j\neq i}^{n} \frac{\lambda_j}{\lambda_j-\lambda_i}\right] \tag{4.3}$$

$$\lambda_{n-1} p_{n-1}(t) = \sum_{i=0}^{n-1} \lambda_i e^{-\lambda_i t}\left[\prod_{j=0,j\neq i}^{n-1} \frac{\lambda_j}{\lambda_j-\lambda_i}\right]. \tag{4.4}$$

Adding together (4.3) and (4.4) we have

$$-\lambda_n e^{-\lambda_n t}\left[\prod_{j=0,j\neq n}^{n} \frac{\lambda_j}{\lambda_j-\lambda_n}\right] + \sum_{i=0}^{n-1} \lambda_i e^{-\lambda_i t}\left[\prod_{j=0,j\neq i}^{n-1} \frac{\lambda_j}{\lambda_j-\lambda_i}\left\{1-\frac{\lambda_n}{\lambda_n-\lambda_i}\right\}\right].$$

The summation is equal to

$$-\sum_{i=0}^{n-1} \frac{\lambda_i^2}{\lambda_n-\lambda_i} e^{-\lambda_i t}\left[\prod_{j=0,j\neq i}^{n-1} \frac{\lambda_j}{\lambda_j-\lambda_i}\right] = -\frac{1}{\lambda_n}\sum_{i=0}^{n-1} \lambda_i^2 e^{-\lambda_i t}\left[\prod_{j=0,j\neq i}^{n} \frac{\lambda_j}{\lambda_j-\lambda_i}\right]$$

hence (4.3) + (4.4) is equal to

$$\begin{aligned} &-\lambda_n e^{-\lambda_n t}\left[\prod_{j=0,j\neq n}^{n} \frac{\lambda_j}{\lambda_j-\lambda_n}\right] - \frac{1}{\lambda_n}\sum_{i=0}^{n-1} \lambda_i^2 e^{-\lambda_i t}\left[\prod_{j=0,j\neq i}^{n} \frac{\lambda_j}{\lambda_j-\lambda_i}\right] \\ &= -\frac{1}{\lambda_n}\sum_{i=0}^{n} \lambda_i^2 e^{-\lambda_i t}\left[\prod_{j=0,j\neq i}^{n} \frac{\lambda_j}{\lambda_j-\lambda_i}\right], \end{aligned}$$

which is exactly (4.2); this completes the exercise.

Exercise 4- 39: Consider a linear birth process $N = (N_t)_{t\geq 0}$ with birth rates given by $\lambda_n = n\lambda$, for $n \in \mathbb{N}$, $\lambda > 0$. Assume that $N_0 = 1$. Determine whether or not this birth process explodes and justify your answer.

Solution: Using the fact that the harmonic series diverges, we have that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \frac{1}{\lambda}\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

From Theorem 4.3.1 from the lecture notes we can conclude that the probability that explosion occurs is equal to zero.

4.5 Prerequisites: Lecture 22

Exercise 4- 40: Consider a population of N individuals consisting at time 0 of one 'infective' and $N-1$ 'susceptibles'. The process changes only by susceptibles becoming infective. We assume that this process can be modelled as a birth process. If, at some time t, there are i infectives, then, for each susceptible, there is a probability of $i\lambda\delta + o(\delta)$ of becoming infective in $(t, t+\delta]$ for $\lambda, \delta > 0$.

(a) If we consider the event of becoming an infective as a birth, what is the birth rate λ_i of the process, when there are i infectives?

(b) Let T denote the time to complete the epidemic, i.e. the first time when all N individuals are infective.

1. Derive $\mathrm{E}(T)$ (without using any type of generating functions).
2. Show that the Laplace transform of T is given by
$$\mathrm{E}[e^{-sT}] = \prod_{i=1}^{N-1}\left(\frac{\lambda_i}{\lambda_i + s}\right), \qquad \text{for } s \geq 0.$$
3. Derive $\mathrm{E}(T)$ by using the Laplace transform given in (2.).

You may leave your solution in (1.) and (3.) as a sum.

Solution:

(a) If there are i infectives, then there are $N-i$ susceptibles and hence the birth rate is given by $\lambda_i = (N-i)i\lambda$ if $i = 1, \ldots, N-1$ and 0 otherwise. We can justify the rates as follows: For $t \geq 0$, $\delta > 0$, $i, m \in \mathbb{N}_0$ (and $m \leq N-i$):

$$\mathrm{P}(N_{t+\delta} = i+m | N_t = i) = \binom{N-i}{m}(i\lambda\delta)^m(1-i\lambda\delta)^{N-i-m} + o(\delta)$$
$$= \begin{cases} (1-i\lambda\delta)^{N-i} + o(\delta), & \text{if } m = 0 \\ (N-i)i\lambda\delta(1-i\lambda\delta)^{N-i-1} + o(\delta), & \text{if } m = 1 \\ o(\delta), & \text{if } m > 1. \end{cases}$$
$$= \begin{cases} \sum_{k=0}^{N-i}\binom{N-i}{k}(-i\lambda\delta)^k + o(\delta), & \text{if } m = 0 \\ (N-i)i\lambda\delta\sum_{i=0}^{N-i-1}\binom{N-i-1}{i}(-\lambda\delta)^i + o(\delta), & \text{if } m = 1 \\ o(\delta), & \text{if } m > 1. \end{cases}$$
$$= \begin{cases} 1-(N-i)i\lambda\delta + o(\delta), & \text{if } m = 0 \\ (N-i)i\lambda\delta + o(\delta), & \text{if } m = 1 \\ o(\delta), & \text{if } m > 1. \end{cases}$$

(b)

1. Let X_i be the time spent in state i (where i denotes the number of infectives). (Note that if you would like to choose the same notation as in the lecture notes, then you can use the holding time notation associated with particular states, here we have $X_i = H_{|i}$.) Then we have that the time to complete the epidemic is
$$T = X_1 + \cdots + X_{N-1},$$
where the X_i are independent of each other with $X_i \sim \mathrm{Exp}(\lambda_i)$. By the linearity of the expectation,
$$\mathrm{E}[T] = \sum_{i=1}^{N-1}\mathrm{E}(X_i) = \sum_{i=1}^{N-1}\frac{1}{\lambda_i} = \sum_{i=1}^{N-1}\frac{1}{(N-i)i\lambda}.$$
2. Let $s \geq 0$. Using the notation from (1.), the Laplace transform of X_i is given by
$$\mathrm{E}(e^{-sX_i}) = \int_0^\infty e^{-sx}\lambda_i e^{-\lambda_i x}dx = \lambda_i\int_0^\infty e^{-(s+\lambda_i)x}dx = \frac{\lambda_i}{\lambda_i + s}.$$
Hence
$$\mathrm{E}[e^{-sT}] = \mathrm{E}[e^{-s\sum_{i=1}^{N-1}X_i}] \overset{\text{independence of } X_is}{=} \prod_{i=1}^{N-1}\mathrm{E}[e^{-sX_i}] = \prod_{i=1}^{N-1}\left(\frac{\lambda_i}{\lambda_i + s}\right).$$

3. To compute the expectation, we calculate the logarithm of the Laplace transform and use the fact that

$$\mathrm{E}[T] = -\frac{d}{ds}\left[\log\left\{\mathrm{E}[e^{-sT}]\right\}\right]\Big|_{s=0}.$$

To see this, note that, by the chain rule for differentiation (assuming we can interchange the expectation and the derivative),

$$\frac{d}{ds}\left[\log\left\{\mathrm{E}[e^{-sT}]\right\}\right] = \frac{1}{\mathrm{E}[e^{-sT}]} \times \frac{d}{ds}\mathrm{E}[e^{-sT}] = \frac{\mathrm{E}[(-T)e^{-sT}]}{\mathrm{E}[e^{-sT}]}.$$

Hence

$$-\frac{d}{ds}\left[\log\left\{\mathrm{E}[e^{-sT}]\right\}\right]\Big|_{s=0} = -\frac{\mathrm{E}(-Te^{0})}{\mathrm{E}(e^{0})} = \mathrm{E}(T).$$

Here

$$\log\left\{\mathrm{E}[e^{-sT}]\right\} = \sum_{i=1}^{N-1}\log\left(\frac{\lambda_i}{\lambda_i+s}\right),$$

and

$$\frac{d}{ds}\log\left\{\mathrm{E}[e^{-sT}]\right\} = \sum_{i=1}^{N-1}\frac{(\lambda_i+s)}{\lambda_i}\cdot\frac{(\lambda_i+s)\cdot 0-\lambda_i\cdot 1}{(\lambda_i+s)^2} = -\sum_{i=1}^{N-1}\frac{1}{(\lambda_i+s)}.$$

Hence,

$$\mathrm{E}[T] = \sum_{i=1}^{N-1}\frac{1}{\lambda_i} = \sum_{i=1}^{N-1}\frac{1}{(N-i)i\lambda}.$$

The reason for taking the log of the Laplace transform was that it made the computation easier in the sense that we did not need to differentiate a product consisting of $N-1$ components. Alternatively, we could have argued that

$$-\frac{d}{ds}\left[\mathrm{E}[e^{-sT}]\right]\Big|_{s=0} = -\left[\mathrm{E}[e^{-sT}](-T)\right]\Big|_{s=0} = \left[\mathrm{E}[e^{-sT}](T)\right]\Big|_{s=0} = \mathrm{E}(T).$$

Here, we have

$$\begin{aligned}\frac{d}{ds}\prod_{i=1}^{N-1}\left(\frac{\lambda_i}{\lambda_i+s}\right) &= \sum_{i=1}^{N-1}\left(\frac{d}{ds}\frac{\lambda_i}{\lambda_i+s}\right)\prod_{j=1,j\neq i}^{N-1}\frac{\lambda_j}{\lambda_j+s}\\ &= \sum_{i=1}^{N-1}\left((-1)\frac{\lambda_i}{(\lambda_i+s)^2}\right)\prod_{j=1,j\neq i}^{N-1}\frac{\lambda_j}{\lambda_j+s}.\end{aligned}$$

Plugging in $s=0$ and multiplying by (-1) leads to

$$\mathrm{E}(T) = -\sum_{i=1}^{N-1}\left((-1)\frac{\lambda_i}{(\lambda_i)^2}\right)\prod_{j=1,j\neq i}^{N-1}\frac{\lambda_j}{\lambda_j} = \sum_{i=1}^{N-1}\frac{1}{\lambda_i} = \sum_{i=1}^{N-1}\frac{1}{(N-i)i\lambda}.$$

Exercise 4- 41: A colony of $N>1$ creatures inhabit a planet which has continual daylight, and the pattern of waking and sleeping follows a continuous-time homogeneous Markov chain, more precisely, a birth-death process. The probability that a particular sleeping individual awakes during a time interval of length $[t, t+\delta]$ is $\beta\delta + o(\delta)$, and the probability that a particular awake individual falls asleep during a time interval $[t, t+\delta]$ of length is $\nu\delta + o(\delta)$. Assume that individuals behave independently of each other. We are interested in the number of individuals awake at time t.

(a) Find the generator matrix.

(b) Find the stationary distribution.

Consider the 2-state Markov chain (with states s sleep and w wake) for one individual with transition matrix

$$\mathbf{P}_t = \begin{pmatrix} 1 - p_{sw}(t) & p_{sw}(t) \\ 1 - p_{ww}(t) & p_{ww}(t) \end{pmatrix}.$$

(c) Write down the generator for this 2-state process.

(d) Calculate $p_{ww}(t)$ and $p_{sw}(t)$ using the forward equations.

(e) If $X_{m,t}$ denotes the number awake at time t given there are $m < N$ awake at time 0, what is $\mathrm{E}[X_{m,t}]$?

Solution:

(a) For each individual we have:

$$\mathrm{P}(\text{it wakes up in } (t, t+\delta] | \text{it was asleep at } t) = \beta\delta + o(\delta),$$
$$\mathrm{P}(\text{it falls asleep in } (t, t+\delta] | \text{it was awake at } t) = \nu\delta + o(\delta).$$

There are N individuals, thus if there are i awake, there are $N - i$ asleep; the 'birth' and 'death' rates are

$$\lambda_i = (N-i)\beta, \qquad \mu_i = i\nu,$$

for $i = 0, \ldots, N$. These rates can we justified as follows: For $t \geq 0$, $\delta > 0$, $i \in \{0, \ldots, N\}$, $m \in \mathbb{Z}$ (such that $i + m \in \{0, \ldots, N\}$):

$$\begin{aligned}
&\mathrm{P}(X_{t+\delta} = i + m | X_t = i) \\
&= \begin{cases} \binom{N-i}{1}(\beta\delta)^1(1-\beta\delta)^{N-i-1}(1-\nu\delta)^i + o(\delta), & \textit{if } m = 1 \\ (1-\beta\delta)^i\binom{i}{1}(\nu\delta)^1(1-\nu\delta)^{i-1} + o(\delta), & \textit{if } m = -1 \\ o(\delta), & \textit{if } |m| > 1. \end{cases} \\
&= \begin{cases} (N-i)\beta\delta + o(\delta), & \textit{if } m = 1 \\ i\nu\delta + o(\delta), & \textit{if } m = -1 \\ o(\delta), & \textit{if } |m| > 1. \end{cases}
\end{aligned}$$

Thus the generator matrix is given by

$$\mathbf{G} = \begin{pmatrix} -N\beta & N\beta & 0 & 0 & \cdots \\ \nu & -\nu - \beta(N-1) & \beta(N-1) & 0 & \cdots \\ 0 & 2\nu & -2\nu - \beta(N-2) & \beta(N-2) & \\ & \ddots & \ddots & \ddots & \\ & & (N-1)\nu & -(N-1)\nu - \beta & \beta \\ & & & N\nu & -N\nu \end{pmatrix}.$$

(b) For the stationary distribution, solve $\boldsymbol{\pi}\mathbf{G} = \mathbf{0}$, from notes, for a general birth and death process we have

$$\begin{aligned}\pi_n &= \frac{\lambda_0 \dots \lambda_{n-1}}{\mu_1 \dots \mu_n}\pi_0, \quad n \in \mathbb{N} \\ &= \frac{(N\beta)(\beta(N-1)) \times \beta(N-n+1)}{\nu(2\nu)\dots(n\nu)}\pi_0 \\ &= \frac{\beta^n}{\nu^n}\binom{N}{n}\pi_0.\end{aligned}$$

Also, since $\sum_{n=0}^{N}\pi_n = 1$, giving

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{N}\frac{\beta^n}{\nu^n}\binom{N}{n}}.$$

(c) For one individual

$$\mathbf{G} = \begin{pmatrix} -\beta & \beta \\ \nu & -\nu \end{pmatrix}.$$

(d) From the forward equations $\mathbf{P}_t' = \mathbf{P}_t\mathbf{G}$, it follows

$$\begin{aligned}\frac{d}{dt}(1 - p_{sw}(t)) &= -\beta(1 - p_{sw}(t)) + \nu p_{sw}(t) \\ \Leftrightarrow -\frac{d}{dt}p_{sw}(t) &= p_{sw}(t)(\beta+\nu) - \beta \\ \Leftrightarrow \frac{d}{dt}p_{sw}(t) + p_{sw}(t)(\beta+\nu) &= \beta\end{aligned}$$

Using the integrating factor $M(x) = \exp((\beta+\nu)x)$ and since $p_{sw}(0) = 0$, the constant $C = 0$ and thus

$$p_{sw}(t) = \int_0^t \beta\exp((\beta+\nu)u)du \cdot \exp(-(\beta+\nu)t) = \frac{\beta}{\beta+\nu}(1 - e^{-(\beta+\nu)t}).$$

In a similar manner as before

$$\frac{d}{dt}(1 - p_{ww}(t)) = -\beta(1 - p_{ww}(t)) + \nu p_{ww}(t),$$

with the boundary condition $p_{ww}(0) = 1$, so

$$p_{ww}(t) = \frac{\beta + \nu e^{-(\beta+\nu)t}}{\beta+\nu}.$$

(e) $X_{m,t}$ denotes the number of individuals awake at time $t > 0$, given that at time 0 there were $m < N$ individuals awake. Recall that all individuals behave independently of each other. We get that

$$X_{m,t} = Y_{m,t} + Z_{m,t},$$

where

$$Y_{m,t} \sim \text{Binomial}(m, p_{ww}(t)), \qquad Z_{m,t} \sim \text{Binomial}(N-m, p_{sw}(t)).$$

Hence, we have two random variables with Binomial distribution, where the probability of success is the probability that at time t, the individual will be awake. We have that

$$\mathrm{E}(Y_{m,t}) = mp_{ww}(t), \qquad \mathrm{E}(Z_{m,t}) = (N-m)p_{sw}(t).$$

Altogether we get

$$\begin{aligned}\mathrm{E}[X_{m,t}] &= mp_{ww}(t) + (N-m)p_{sw}(t) \\ &= me^{-(\beta+\nu)t} + \frac{N\beta}{\beta+\nu}(1-e^{-(\beta+\nu)t}).\end{aligned}$$

Exercise 4- 42: A population member alive at t dies during $(t, t+\delta)$ with probability $\mu\delta + o(\delta)$, independently of other population members. The population changes size only from the death of population members (there are no births, emigration or immigration). We assume that the population size can be modelled as a death process. The initial population size is n_0. Let T be the time at which the population dies out; i.e. $T = \min\{t \geq 0 : N_t = 0\}$. By considering the times between successive changes in population size find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$.

Solution: Let T_n be the first time the population drops to size n or below, i.e.

$$T_n = \min\{t \geq 0 : N_t \leq n\}$$

(for a non-negative integer $n \leq n_0$). Then $N_{T_0} = 0$ and $T = T_0$. We write T as

$$T = (T_0 - T_1) + (T_1 - T_2) + \cdots (T_{n_0-1} - T_{n_0}),$$

where $T_{n_0} = 0$. Then $T = X_1 + \cdots + X_{n_0}$, where $X_i = T_{i-1} - T_i$ is the time the population has size i before dropping to size $i-1$. We have that $X_1, \ldots, X_{n_0}$ are independent with $X_i \sim \mathrm{Exp}(\mu i)$ for $i \in \{1, \ldots, n_0\}$. Using standard results in probability and the moments of an exponential distribution it follows that

$$\begin{aligned}\mathrm{E}(T) &= \frac{1}{\mu}\left[1 + \frac{1}{2} + \cdots + \frac{1}{n_0}\right], \\ \mathrm{Var}(T) &= \frac{1}{\mu^2}\left[1 + \frac{1}{4} + \cdots + \frac{1}{n_0^2}\right].\end{aligned}$$