# 5 Problem sheet 5: Brownian motion

## 5.1 Prerequisites: Lecture 23

**Exercise 5- 43:** What is the autocovariance of a standard Brownian motion? I.e. compute $\text{Cov}(W_t, W_s)$, for $s, t \geq 0$.

**Solution:** We know that

$$\text{Cov}(W_t, W_s) = \text{E}(W_t W_s) - \underbrace{\text{E}(W_t)}_{=0}\underbrace{\text{E}(W_s)}_{=0} = \text{E}(W_t W_s).$$

Without loss of generality, we assume that $0 \leq s \leq t$. Then

$$\begin{aligned}\text{Cov}(W_t, W_s) &= \text{E}(W_t W_s) \\ &= \text{E}(W_s^2 - W_s^2 + W_t W_s) = \text{E}(W_s^2 + (W_t - W_s)W_s) \\ &= \text{E}(W_s^2) + \text{E}(W_t - W_s)\text{E}(W_s) \quad \text{due to the independence of the increments,} \\ &= \text{Var}(W_s) + 0 = s.\end{aligned}$$

In general, we have $\text{Cov}(W_t, W_s) = \min(s, t)$.

**Exercise 5- 44:** *Definition:* A stochastic process $(X_t)_{t\geq 0}$ is called a *Gaussian process* if $(X_{t_1}, \dots, X_{t_n})$ has multivariate normal distribution for all $t_1, \dots, t_n, n \in \mathbb{N}$.

Using the famous Cramer-Wold device, we have the following result: A random vector $\mathbf{X} = (X_1, \dots, X_n)^\top \in \mathbb{R}^n$ follows a multivariate normal distribution if $\mathbf{t}^\top \mathbf{X}$ follows a univariate normal distribution for all $\mathbf{t} = (t_1, \dots, t_n)^\top \in \mathbb{R}^n$.

Show the following results:

1. A Brownian motion $B$ is a Gaussian process with zero mean and $\text{Cov}(B_t, B_s) = \min(t, s)$ for $s, t \geq 0$.
2. A Gaussian process $(X_t)_{t\geq 0}$ starting at zero, having continuous sample paths and having mean zero and covariance given by $\text{Cov}(X_t, X_s) = \min(s, t)$ for all $s, t \geq 0$, is a Brownian motion.

**Solution:**

1. Consider any $n \in \mathbb{N}$ and any $t_1 < \cdots < t_n$. We need to show (using the Cramer-Wold device) that for any constants $a_1, \dots, a_n$, we have that $\sum_{i=1}^n a_i B_{t_i}$ follows a univariate normal distribution. Recall that the increments $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ of the Brownian motion are independent and normally distributed. Using the telescoping sum and setting $B_{t_0} = B_0 = 0$, we have $B_{t_i} = \sum_{k=1}^i (B_{t_k} - B_{t_{k-1}})$, for $i = 1, \dots, n$. Hence, we write

$$\sum_{i=1}^n a_i B_{t_i} = \sum_{i=1}^n a_i \sum_{k=1}^i (B_{t_k} - B_{t_{k-1}}) = \sum_{k=1}^n (B_{t_k} - B_{t_{k-1}}) \sum_{i=k}^n a_i,$$

which is a linear combination of independent Gaussian random variables and is hence normally distributed. By the Cramer-Wold device stated above, we deduce that $(B_{t_1}, \dots, B_{t_n})$ has a multivariate Gaussian distribution.

We note that $\text{E}(B_t) = 0$ and the covariance has been derived in Exercise 5- 43 above.

2. We only need to check the that the increments are independent and stationary with normal distribution (where the variance depends on the corresponding interval length).

   Consider any $n \in \mathbb{N}$ and any $t_1 < \cdots < t_n$. Since $B$ is a Gaussian process, $(B_{t_1}, \ldots, B_{t_n})$ has multivariate Gaussian distribution. We want to show that $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ has multivariate normal distribution. To see that, consider any constants $b_i, \ldots, b_n$, set $B_{t_0} = B_0 = 0$, and write

$$\sum_{i=1}^{n} b_i(B_{t_i} - B_{t_i-1}) = \sum_{i=1}^{n} b_i B_{t_i} - \sum_{i=1}^{n} b_i B_{t_{i-1}} = \sum_{i=1}^{n} b_i B_{t_i} - \sum_{i=0}^{n-1} b_{i+1} B_{t_i}$$
$$= \sum_{i=1}^{n-1} (b_i - b_{i+1}) B_{t_i} + b_n B_{t_n} - b_1 B_0 = \sum_{i=1}^{n-1} (b_i - b_{i+1}) B_{t_i} + b_n B_{t_n},$$

   which is a linear combination of jointly Gaussian random variables and hence normally distributed. Moreover, the increments have mean

$$\mathrm{E}(B_{t_i} - B_{t_{i-1}}) = \mathrm{E}(B_{t_i}) - \mathrm{E}(B_{t_{i-1}}) = 0 - 0 = 0,$$

   and variance

$$\mathrm{Var}(B_{t_i} - B_{t_{i-1}}) = \mathrm{Var}(B_{t_i}) + \mathrm{Var}(B_{t_{i-1}}) - 2\mathrm{Cov}(B_{t_i}, B_{t_{i-1}}) = t_i + t_{i-1} - 2t_{i-1} = t_i - t_{i-1}.$$

   Also, for $i, k \in \{1, \ldots, n\}, i \neq k$, w.l.o.g. suppose that $i < k$,

$$\mathrm{Cov}(B_{t_i} - B_{t_{i-1}}, B_{t_k} - B_{t_{k-1}}) = \min(t_i, t_k) - \min(t_i, t_{k-1}) - \min(t_{i-1}, t_k) + \min(t_{i-1}, t_{k-1})$$
$$= t_i - t_i - t_{i-1} + t_{i-1} = 0.$$

   So, we found that the increments are multivariate Gaussian and uncorrelated, hence they are independent. Also, $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$ for $i = 1, \ldots, n$ (where $t_0 := 0$), which implies stationarity.

**Exercise 5- 45:** Let $W$ denote a standard Brownian motion. Show that

$$Y_t = W_1 - W_{1-t}, \quad 0 \leq t \leq 1,$$

is a standard Brownian motion for $0 \leq t \leq 1$. [Time reversal]

**Solution:**

1. $Y_0 = W_1 - W_1 = 0$.

2. Let $0 \leq t < T < s < S \leq 1$. Then $1 - S < 1 - s < 1 - T < 1 - t$. Since the increments of $W$ are independent, we can deduce that the increments of $Y$ given by

$$Y_T - Y_t = W_1 - W_{1-T} - (W_1 - W_{1-t}) = W_{1-t} - W_{1-T},$$

   and

$$Y_S - Y_s = W_{1-s} - W_{1-S},$$

   are also independent. We have shown the independence of two increments; using similar arguments, one can show that $n$ increments are independent (for $n \in \mathbb{N}$).

3. Since $W$ has stationary increments, we obtain, for any $0 \leq t < T \leq 1$,

$$Y_T - Y_t = W_{1-t} - W_{1-T} \stackrel{d}{=} W_{1-t-(1-T)} = W_{T-t},$$
$$Y_{T-t} = W_1 - W_{1-(T-t)} \stackrel{d}{=} W_{1-(1-(T-t))} = W_{T-t}.$$

[Recall that $\stackrel{d}{=}$ stands for equality in distribution. I.e. the random variable on the left hand side has the same distribution as the random variable on the right hand side.]

Hence, $Y_T - Y_t \stackrel{d}{=} Y_{T-t}$, i.e. $Y$ has stationary increments.

4. Since $W$ has Gaussian distributed increments, we have, for any $0 \leq t < T \leq 1$,

$$Y_T - Y_t \stackrel{d}{=} W_{T-t} \sim N(0, T-t).$$

5. Since $W$ has continuous sample paths, $Y$ has also continuous sample paths.

**Exercise 5- 46:** Let $W$ denote a standard Brownian motion. Show that $(Y_t)_{t\geq 0}$ with $Y_0 = 0$ and

$$Y_t = tW_{1/t}, \quad t > 0,$$

is a standard Brownian motion. [Time inversion]
*Hint:* You may assume without proof that $Y$ is continuous at $0$.

**Solution:** We use Exercise 5- 44 (2.) to show that $Y$ is a standard Brownian motion.

1. $Y_0 = 0$ (using the hint).
2. Clearly $(Y_{t_1}, \ldots, Y_{t_n})$ has multivariate normal distribution for all $t_1, \ldots, t_n, n \in \mathbb{N}$ since it is a linear combination of the components of the Gaussian process $W$. So $Y$ is a Gaussian process.
3. $Y$ has continuous sample paths. Note that it is not trivial to show continuity at 0. But we don't have the mathematical tools for proving this in the current course.
4. $\mathrm{E}(Y_t) = t\mathrm{E}(W_{1/t}) = 0$.
5. Finally, we need to check the covariance structure: Let $0 < t < s$, hence $\frac{1}{s} < \frac{1}{t}$. Then

$$\mathrm{Cov}(Y_t, Y_s) = \mathrm{Cov}(tW_{1/t}, sW_{1/s}) = tsCov(W_{1/t}, W_{1/s}) = ts \min\left(\frac{1}{s}, \frac{1}{t}\right) = ts\frac{1}{s} = t = \min(s,t).$$

Also,

$$\mathrm{Cov}(Y_0, Y_t) = 0, \text{ for any } t \geq 0.$$

From Exercise 5- 44 (2.), we conclude that $Y$ is indeed a standard Brownian motion.

**Exercise 5- 47:** Let $B = (B_t)_{t\geq 0}$ denote a standard Brownian motion. Let $a > 0$ denote a deterministic constant. Show that $W = (W_t)_{t\geq 0}$ with $W_t = aB_{t/a^2}$ is a standard Brownian motion.

**Solution:**

1. $W_0 = aB_{0/a^2} = aB_0 = 0$, since $B_0 = 0$.
2. Choose any $n \in \mathbb{N}$ and any $0 \le t_1 < t_2 < \cdots < t_n$. Set $s_i = t_i/a^2$, $i = 1, \dots, n$. Then $0 \le s_1 < s_2 < \cdots < s_n$. Since $B$ has independent increments, we know that $B_{s_1}, B_{s_2} - B_{s_1}, \dots, B_{s_n} - B_{s_{n-1}}$ are independent. Multiplying the increments by the deterministic constant $a$, we get that $W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$, are independent.
3. For $0 \le s < t$, we have that $W_t - W_s = a(B_{t/a^2} - B_{s/a^2}) \stackrel{\mathrm{d}}{=} aB_{(t-s)/a^2} = W_{t-s}$, since $B$ has stationary increments.
4. For $0 \le s < t$, we have that $W_t - W_s \stackrel{\mathrm{d}}{=} aB_{(t-s)/a^2} \sim N(0, t-s)$, since $B_{(t-s)/a^2} \sim N(0, (t-s)/a^2)$.
5. The continuity of the sample paths $t \mapsto W_t$ is a direct consequence of the continuity of the paths of $B$.

**Exercise 5- 48:** Let $W = (W_t)_{t \ge 0}$ and $B = (B_t)_{t \ge 0}$ be independent standard Brownian motions. Let $\alpha, \beta \in \mathbb{R}$. Let

$$Y_t = \alpha W_t + \beta B_t,$$

for $t \ge 0$. Derive sufficient conditions on $\alpha, \beta$ to ensure $Y$ is a standard Brownian motion.

**Solution:** We claim that if $\alpha^2 + \beta^2 = 1$, then $Y$ is a standard Brownian motion. (Then we have: $\mathrm{Var}(Y_t) = \alpha^2 t + \beta^2 t = t$.)

We check the conditions stated in Exercise 5- 44 (2.). $Y$ is the linear combination of two independent Gaussian processes and hence a Gaussian process. We have that $Y_0 = 0$ and that its sample paths are continuous since its a linear combination of two Brownian motions. Also, we have $\mathrm{E}(Y_t) = 0$ and, for all $s, t \ge 0$,

$$\begin{aligned}\mathrm{Cov}(Y_s, Y_t) &= \mathrm{Cov}(\alpha W_s + \beta B_s, \alpha W_t + \beta B_t)\\ &= \alpha^2 \mathrm{Cov}(W_s, W_t) + \alpha\beta \mathrm{Cov}(W_s, B_t) + \beta\alpha \mathrm{Cov}(B_s, W_t) + \beta^2 \mathrm{Cov}(B_s, B_t)\\ &= (\alpha^2 + \beta^2) \min(s, t) = \min(s, t),\end{aligned}$$

if $\alpha^2 + \beta^2 = 1$, where we used the independence of $B$ and $W$.

# References

Grimmett, G. R. & Stirzaker, D. R. (2001*a*), *One Thousand Exercises in Probability*, Oxford University Press, New York.

Grimmett, G. R. & Stirzaker, D. R. (2001*b*), *Probability and random processes*, third edn, Oxford University Press, New York.