GALOIS THEORY
Worksheet 1

Q 1. This question is designed to get you up to speed with arithmetic in $\mathbb{Q}[X]$.

(a) Find the quotient and remainder when $X^5 + X + 1$ is divided by $X^2 + 1$.

(b) Find the remainder when $X^{2019} + 32X^{53} + 8$ is divided by $X - 1$.

(c) Find polynomials $s(X)$ and $t(X)$ such that

$$(2X^3 + 2X^2 + 3X + 2)s(X) + (X^2 + 1)t(X) = 1$$

How did I know that these polynomials were coprime?

(d) Find a hcf for X^4+4 and X^3-2X+4. Express it as $(X^4+4)a(X)+(X^3-2X+4)b(X)$.

(e) Find polynomials $\lambda(X)$ and $\mu(X)$ such that $(1 + X)\lambda(X) + (X^3 - 2)\mu(X) = 1$.

Q 2. Write $\xi = \sqrt[3]{2}$. This question is about understanding the field operations in $\mathbb{Q}(\xi)$ explicitly.

(a) Find rational numbers a, b and c such that $a + b\xi + c\xi^2 = 1/(1 + \xi)$. [*Hint*: use Part (e) of the previous question.]

(b) Fix rational numbers A, B. Find formulas for rational numbers a, b and c, depending on A, B, such that $a + b\xi + c\xi^2 = 1/(A + B\xi + \xi^2)$.

Q 3. Prove that if $f, g \in K[X]$ and at least one is non-zero, and if s, t are both hcf's of f and g, then $s = \lambda t$ for some $\lambda \in K^\times$.

Q 4. (a) We know that whether or not a polynomial is irreducible depends on which field it's considered as being over: for example X^2+1 is irreducible in $\mathbb{Q}[X]$ but not in $\mathbb{C}[X]$. But show that the notion of divisibility does not depend on such issues. More precisely show that if $K \subseteq L$ are fields, if $f, g \in K[X]$, and if $f \mid g$ in $L[X]$ then $f \mid g$ in $K[X]$.

(b) Is it true that if $f, g \in \mathbb{Z}[X]$ and $f \mid g$ in $\mathbb{Q}[X]$ then $f \mid g$ in $\mathbb{Z}[X]$? [*Hint*: No.] Is it true under the extra assumption that f is monic? [*Hint*: Yes.]