GALOIS THEORY
Worksheet 2

Q 1. (a) Prove that if $n \in \mathbb{Z}$ and $\sqrt{n} \notin \mathbb{Z}$ then $\sqrt{n} \notin \mathbb{Q}$.

(b) Prove that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. What is the minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$?

(c) Use the tower law to prove that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$. Write down a basis for the $\mathbb{Q}$-vector space $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Q 2. (a) Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

[*Hint*: the smallest subfield of the complex numbers containing $\mathbb{Q}$ and $\sqrt{2} + \sqrt{3}$ must contain loads of other things too: write some of them down.]

(b) Deduce that $X^4 - 10X^2 + 1$ is irreducible over $\mathbb{Q}$.

Q 3. Is $\sqrt{10} \in \mathbb{Q}(\sqrt{6}, \sqrt{15})$?

Q 4. (a) Prove that if $K \subseteq L$ is a finite extension of fields and V/L is a finite-dimensional vector space then $\dim_K(V) = [L : K] \dim_L(V)$.

(b) Prove that if $K \subseteq L \subseteq E$ and $[E : K] = [L : K]$ is finite, then $L = E$.

Q 5. In this question, $K \subset L$ is a *finite* field extension.

(a) Let R be a ring, $K \subset R \subset L$. Show that, actually, R is a field.

(b) For a set $S \subset L$, show that there is a smallest subfield of L that contains K and S. Denote this field by $K(S)$. Show that there is a finite subset of $T \subset S$ such that $K(T) = K(S)$ and that, in fact $K(S) = K[T]$, where $K[T]$ is the ring of polynomial expressions in T with coefficients in K.

(c) Let now $K \subset F_1 \subset L$ and $K \subset F_2 \subset L$ be subfields. Show that there is a smallest subfield of L that contains F_1 and F_2. Denote this field by F_1F_2. Show that F_1F_2 is the set of finite sums:

$$\sum_{\text{finite}} \lambda_i \mu_i \quad \text{where, for all } i, \quad \lambda_i \in F_1,\ \mu_i \in F_2$$