GALOIS THEORY
Solutions to Worksheet 5

A 1. Write $K = \mathbb{Q}\left(\sqrt{2}, \sqrt{-3}, \sqrt[3]{5}\right)$. Observe that

$$X^3 - 5 = (X - \sqrt[3]{5})(X - \omega\sqrt[3]{5})(X - \omega^2\sqrt[3]{5})$$

where

$$\omega = \frac{-1 + \sqrt{-3}}{2}$$

is a primitive cube root of unity. It follows from this that K is the splitting field of the polynomial

$$f(X) = (X^2 - 2)(X^3 - 5) \in \mathbb{Q}[X]$$

indeed the polynomial splits completely in K and K is generated by the roots (if $\sqrt[3]{5}$ and $\omega\sqrt[3]{5}$ are both in F, then clearly ω is also in F). Hence $\mathbb{Q} \subset L$ is a normal extension.

Now let us count degrees. First, let us state that $\sqrt{2} \notin \mathbb{Q}$, hence $\left[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}\right] = 2$. Next consider the field $L = \mathbb{Q}(\sqrt{-3}, \sqrt{2})$. It is clear that, say, $\sqrt{-3} \notin \mathbb{Q}(\sqrt{2})$—for example, $\sqrt{-3}$ is purely imaginary while $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$. If you don't like this, suppose for a contradiction that $\sqrt{-3} \in \mathbb{Q}(\sqrt{2})$, that is there exist rational numbers $x, y \in \mathbb{Q}$ such that

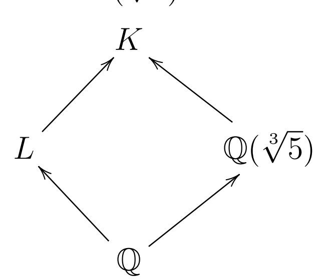
$$-3 = (x + y\sqrt{2})^2 = x^2 + 2y^2 + 2xy\sqrt{2}$$

since this is an identity in a 2-dimensional vector space over $\mathbb{Q}$ with basis $1, \sqrt{2}$ we must have either $x = 0$ or $y = 0$. If $y = 0$, then $x^2 = -3$, $x \in \mathbb{Q}$ leads easily to a contradiction. If $x = 0$ then $-3 = 2y^2$. Writing $y = p/q$ with p, q coprime integers, we have

$$-3q^2 = 2p^2$$

and we easily get a contradiction working $2-$ or $3-$adically.[1] By a simple application of the tower law then $[L : \mathbb{Q}] = 4$.

Finally let us consider our field $K = L(\sqrt[3]{5})$ and the diagram of field extensions:



[1] I am deliberately avoiding reaching a contradiction by means of the order structure of the rationals: the left hand side is negative, the right hand side is positive. This would be reproducing the argument in terms of imaginary numbers that we wanted to avoid.

I claim that $X^3 - 5$ is irreducible in $L[X]$ and hence $[K : L] = 3$ and then $[K : \mathbb{Q}] = [K : L][L : \mathbb{Q}] = 3 \times 4 = 12$. Indeed if $X^3 - 5$ were not irreducible in $L[X]$ then it would have a root $\alpha \in L$; and then from $\mathbb{Q} \subset \mathbb{Q}(\alpha) \subset L$ we would conclude from the tower law that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ divides $[L : \mathbb{Q}] = 4$, a contradiction. Hence $[K : \mathbb{Q}] = 12$.

A 2. (a) If $[F : E] = 2$ let $\alpha \in F \setminus E$, then consider the tower of field extensions $E \subset E(\alpha) \subset F$. As a simple consequence of the tower law we get that $F = E(\alpha)$. The minimal polynomial of α over E has degree 2:

$$f(X) = X^2 + aX + b \in E[X]$$

and $X - \alpha$ divides $f(X)$ in $F[X]$ hence $f(X)$ splits completely in F, hence F is the splitting field of $f(X)$ hence $E \subset F$ is a normal extension.

(b) Suppose that $H \leq G$ has index two. This means that there are two elements (cosets) in the quotient set $X = H\backslash G$ and also in the quotient set $Y = G/H$. Let $g \in G$ be any element: if $g \in H$ then clearly $g^{-1}Hg = H$, so let us assume that $g \notin H$. It must be the case that $Hg = G \setminus H$ AND $gH = G \setminus H$; therefore $Hg = gH$.[2]

A 3.

$$\begin{aligned}
\mathbb{Q} &\subseteq \mathbb{Q}\left(8^{1/5}\right) \\
&\subseteq \mathbb{Q}\left(8^{1/5}, \sqrt{8^{1/5}+6}\right) \\
&\subseteq \mathbb{Q}\left(8^{1/5}, \sqrt{8^{1/5}+6}, 5^{1/3}\right) \\
&\subseteq \mathbb{Q}\left(8^{1/5}, \sqrt{8^{1/5}+6}, 5^{1/3}, \sqrt[11]{5^{1/3}+\sqrt{8^{1/5}+6}}\right) \\
&\subseteq \mathbb{Q}\left(8^{1/5}, \sqrt{8^{1/5}+6}, 5^{1/3}, \sqrt[11]{5^{1/3}+\sqrt{8^{1/5}+6}}, 9^{1/7}\right)
\end{aligned}$$

A 4. This is not difficult at all. Go back to your notes of the discussion of $X^3 - 2$ at the beginning of the course and make the appropriate minor changes.

[2] You are supposed to "see" that the two parts of the question correspond under the Galois correspondence.