# GALOIS THEORY
# Worksheet 6



**Q 1.** Establish (with proofs) whether the following extensions of $\mathbb{Q}$ are normal or not:

(i) $\mathbb{Q}(\sqrt{6})$;

(ii) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$;

(iii) $\mathbb{Q}(7^{1/3})$;

(iv) $\mathbb{Q}(7^{1/3}, e^{2\pi i/3})$;

(v) $\mathbb{Q}(\sqrt{1+\sqrt{7}})$;

(vi) $\mathbb{Q}(\sqrt{2+\sqrt{2}})$.

**Q 2.** Let $K \subset F \subset L$ be a tower of field extensions. It is immediate from the definition that: If $K \subset L$ is normal, then $F \subset L$ is also normal.

(a) If $K = \mathbb{Q}$, $F = \mathbb{Q}(2^{1/3})$ and $L = \mathbb{Q}(2^{1/3}, \omega)$ with $\omega = e^{2\pi i/3}$, then show that $K \subset L$ is normal, but $K \subset F$ is not normal.

(b) If $K = \mathbb{Q}$, $F = \mathbb{Q}(\sqrt{2})$ and $E = \mathbb{Q}(2^{1/4})$, show that $K \subset F$ and $F \subset L$ are normal, but $K \subset L$ is not normal.

(c) Say $H \subseteq K \subseteq G$ are groups. Prove that if $H$ is normal in $G$ then $H$ is normal in $K$. Give an example of groups with $H$ is normal in $G$ but $K$ not normal in $G$. Now give an example with $H$ normal in $K$, $K$ normal in $G$, but $H$ not normal in $G$. Now wonder whether this is all a coincidence or not.

**Q 3.** Let $p$ be a prime number, and let $\mathbb{Q} \subset K$ be the splitting field of $X^4 - p$. What is $[K : \mathbb{Q}]$? What is the Galois group of $\mathbb{Q} \subset K$?

**Q 4 (†).** In Parts (a) and (c), (d) of this question $K$ is a field of characteristic $\neq 2$.

(a) Suppose that $a, b \in K$ are such that $a$ is a square in $K(\sqrt{b})$.[1] Prove that either $a$ or $ab$ is a square in $K$.

(b) Show by example that the statement in Part (a) is, in general, false if $K$ has characteristic $= 2$.

(c) Let $a, b \in K$ and suppose that $b$ is NOT a square in $K$; let $L = K(\beta)$ with $\beta^2 = b$. Prove that: If one of $a + \beta$, $a - \beta$ is a square in $L$, then so is the other, and deduce that $c = a^2 - b$ is a square in $K$.

(d) Let $a, b \in K$ and set $c = a^2 - b$; suppose that none of $b$, $c$, or $bc$ is a square in $K$. If $L$ is a splitting field of the polynomial:

$$(X^2 - a)^2 - b \in K[X],$$

prove that $[L : K] = 8$.

[*Hint*: use Part (a) and Part (b) repeatedly.]

[1] In general if $K$ is a field we say that $a \in K$ *is a square* iff the polynomial $X^2 - a \in K[X]$ splits into two linear factors, or, equivalently, there exists $\alpha \in K$ such that $\alpha^2 = a$.